

**An Exemplary Mathematics Program  
--U.S. Dept. of Education**

# **Upstream or Downstream? Let Parametric Equations Show You How to Paddle Your Kayak**

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**Note to student:** In the next problem you will explore one of the reasons that parametric equations are so powerful—they allow you to work separately with what happens in the horizontal and vertical directions. This depends, of course, on the fact that was just demonstrated: vertical falling velocity is unaffected by horizontal velocity.

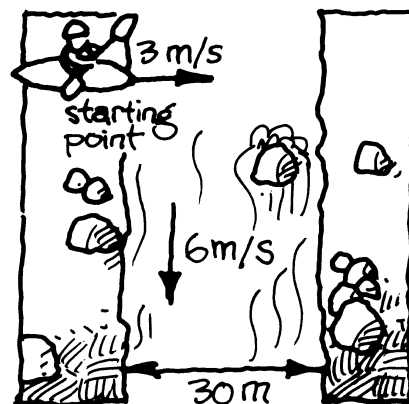
PE-3. While sitting on a pole 256 feet above the ground on a windy day, a gull drops an oyster shell. A wind is blowing the shell sideways, but as we've seen from the nickel demonstration, the height is unaffected. Suppose the wind is blowing the shell sideways at 15 mph, which is 22 feet per second, so  $x = 22t$ . Of course, the shell falls faster and faster as time increases. If we ignore the air resistance as the shell falls, the height  $y$  of the shell at time  $t$  (seconds) is given by  $y = -16t^2 + 256$ , where gravity is given by  $-16t^2$ .

- a) Make a table with three columns as shown below for  $0 \leq t \leq 4$ . The columns represent the time (in seconds), the  $x$ -coordinate of the shell, and the  $y$ -coordinate of the shell. We are essentially graphing two equations of  $t$  in one table. The second column is filled in by using the equation  $x = 22t$ . Fill in this column. The third column is filled in by using the equation  $y = -16t^2 + 256$ . Now fill in the third column.

Time (seconds)	$x = 22t$ x-coordinate (feet)	$y = -16t^2 + 256$ y-coordinate (feet)
0	0	256
0.5	11	252
1	22	240
1.5		

- b) Using a graph with the  $x$ -axis scaled from 0 to 100, and the  $y$ -axis scaled from 0 to 300, plot the position of the shell for times  $t = 0, 0.5, 1, 1.5$ , etc. up to  $t = 4$ .
- c) What sort of curve do you get?
- d) When did the shell hit the ground? How do you know?
- e) The parametric equations for the motion of the shell express  $x$  and  $y$  as functions of  $t$  with  $x = 22t$  and  $y = -16t^2 + 256$ . How would the parametric equations be different if the wind were blowing at 10 ft/s? 40 ft/s?
- f) On the same graph, but using different colored lines, plot the graph of the motion of the shell if the wind is blowing at 30 mph, if it is blowing at 10 mph, if it is blowing at 5 mph, and if it is not blowing at all. Split up the labor of making the tables in your team. Write a few sentences explaining what you notice.

PE-59. Suppose you are in a kayak and want to cross a river. You can paddle your kayak at 3 m/s. You are trying to cross a river flowing at 6 m/s. You aim your kayak directly across the river from your starting point.



- You want to know how far down the river you would land. Figure it out, with some help from your team.
- You also want to know the path of your kayak. To do this, copy the table below onto your paper and fill in the blanks for the 10 seconds it takes you to cross the river.

Time	Distance Across	Distance Upstream
0	0	0
1	3	-6
2		
3		
4		
5		
6		
7		
8		
9		
10		

- What are the units for  $t$ ? For  $x$ ? For  $y$ ?
- Find an equation for  $x$  in terms of  $t$ . Then find another equation for  $y$  in terms of  $t$ .

PE-60. Now suppose you head upstream at an angle of  $40^\circ$ .

- How far did you go horizontally (across the river) in one second
- How far did you go vertically (down the river) in one second? Remember the current!
- Find parametric equations for  $x$  in terms of  $t$  and for  $y$  in terms of  $t$ .
- How long will it take you to get across the river if you paddle in this direction?

PE-61. Now we generalize the situation found in the previous problem.

- You head upstream at an angle  $\theta$ . Express the time  $t$  when the kayak hits the opposite bank in terms of  $\theta$ ?
- Find the distance downstream that your kayak goes in that time.
- Now write the downstream distance  $d$  as a function of  $\theta$ .
- What value of  $\theta$  gives you the minimum downstream distance?
- When you graph  $d$  as a function of  $\theta$ , what does the tangent line to the curve look like at this minimum distance?

# College Preparatory Mathematics 4 (Mathematical Analysis)

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<b>Chapter 1</b> page 1	<b>The Big Game and Radioactivity: Introduction to Models.....IM-1 to IM-157</b> This opening unit provides an introduction to using mathematical models to describe a quantitative relationship. We begin with linear models using Median-Median lines, and include the development of the rules of exponents and exponential models. In addition, the first two programs are written.
<b>Chapter 2</b> page 37	<b>Rocket Launch: Area Between Curves.....RL-1 to RL-206</b> We introduce the notion of area under a curve to the x-axis using a physical situation. In this process we also look at piecewise defined functions and summation notation. The concepts learned here will aid in an understanding of integration which is a major theme in calculus. We continue developing our programming strand by writing programs which find sums and area.
<b>Interlude</b> page 89	<b>Introduction to Logarithms.....IL-1 to IL-62</b> This short unit gives a concentrated development of logarithms and their properties. We first examine the relationship between logs and exponents, then use this relationship to discover the laws of logarithms and to solve exponential equations.
<b>Chapter 3</b> page 107	<b>The Spring Problem: Sinusoidal Functions.....SP-1 to SP-174</b> This is the first of three units focusing on trigonometry. We begin by looking at radian measure and special angles in the unit circle which leads to the development of Parent Graphs for the sine and cosine functions. We then look at transformations of these curves and apply them to periodic motion in applications. We also explore the reciprocal trigonometric functions, basic identities and solving trig equations.
<b>Chapter 4</b> page 149	<b>Algebra for College Math Courses: Big Ideas and Little Tricks.....AL-1 to AL-158</b> This unit will look at some of the standard algebraic techniques needed in college mathematics courses: rationalizing the denominator, etc. We also look at the power of substitution and the effect of simple substitutions on graphs.
<b>Chapter 5</b> page 183	<b>The Next Wave: More Modeling and Trigonometry.....NW-1 to NW-146</b> In this second unit of trigonometry, we build on the earlier ideas from unit 3 to look at more detailed modeling using trigonometric functions. We also look at the inverse trig functions and use these idea to solve more detailed trigonometric equations. The sum and difference formulas are developed and used to find additional identities such as the double angle formulas.
<b>Chapter 6</b> page 217	<b>Running on Empty: Modeling and Statistical Analysis.....RE-1 to RE-145</b> We return to the modeling begun in Unit 1. We introduce the regression line as another linear model, and use it to analyze non-linear data using logs to first straighten it. These techniques allow us to determine how well an exponential or power function fits given data, and when the model breaks down. We use the statistics function on the graphing calculator to bypass difficult arithmetic.

### Units 1-6 Resources

page 251	<b>Skill Builders (Extra Practice)</b> These problems provide concentrated sample problems and additional practice to the basic skills developed in the first half of the course.
page 275	<b>Extra Problems</b> These problems are a random mixture of additional practice that can supplement the current units or act as a review for final exams.
page 285	<b>Appendix A: Quality Control and SAT scores: One Variable Statistics</b> This unit is provided to develop a basic understanding of one variable statistics by looking at measures of center and of spread. Instruction on using the statistics functions on the calculator are given to make these calculations more convenient. We then look at standardization of data using Z-scores in the context of SAT results. The theme problem focuses on sampling techniques used in industry for quality control.
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## Volume II Contains Units 7-13

<b>Chapter 7</b> page 305	<b>The New Pilot: Vectors and More Trigonometry</b> ..... <b>NP-1 to NP-136</b> This trig unit concentrates on applications of trigonometry using the laws of sines and cosines. These ideas lead to the development of vector operations and applications. We look at vectors from both a geometric and algebraic approach developing concepts such as vector addition, unit vectors and the dot product.
<b>Chapter 8</b> page 343	<b>To Infinity and Beyond: Limits</b> ..... <b>LM-1 to LM-213</b> We investigate functions as $x$ approaches a particular value or infinity. Two examples motivate the idea of limits and the concepts are extended to the abstract case. The number $e$ is developed and defined using limits and some famous infinite sums are explored.
<b>Interlude</b> page 389	<b>Introduction to Polar Coordinates</b> ..... <b>PC-1 to PC-77</b> We introduce an alternative coordinate system which is very useful for graphs with a center of symmetry. The relationship between polar coordinates and Cartesian Coordinates is determined. Many beautiful graphs can be created by this alternate method.
<b>Chapter 9</b> page 405	<b>Space Telescope: Rates of Change</b> ..... <b>RC-1 to RC-199</b> Rates of change are introduced for discrete intervals and then limits are used to introduce instantaneous rates of change. The relation between distance traveled and velocity is explored as is the concept of slope of a curve. We bring back the idea of area under a curve and show how it relates to rates of change.
<b>Chapter 10</b> page 447	<b>Conic Sections: Readiness for College Mathematics Texts</b> ..... <b>10-1 to 10-5</b> This unit is written in a form typical of college mathematics text. In addition to learning about the conic forms of circles, ellipses, hyperbolas and parabolas, we show how to read through a college text. We concentrate on showing how to key in on certain ideas and how to follow a proof or derivation covered in a college text.
<b>Chapter 11</b> page 471	<b>Take Me Out to the Ball Game: Parametric Equations</b> ..... <b>PE-1 to PE-69</b> Up to this point we have looked at graphing $y$ as a function of $x$ . In this unit we will look at graphing situations where $x$ and $y$ are both dependent on an independent variable $t$ . Using parametric equations, we can trace the motion of a variety of objects such as the flight of a baseball.
<b>Chapter 12</b> page 491	<b>Linear Transformations: Applications of Matrices</b> ..... <b>LT-1 to LT-77</b> We review matrix operations and use them to solve systems of equations. We extend the applications to include transformations in the plane. In addition, we show how matrices and vectors can be combined to represent more complex transformations in the plane and in 3-dimensional space.
<b>Chapter 13</b> page 513	<b>Where's the Money? Series</b> ..... <b>WM-1 to WM-68</b> Various options for raises in salaries are used to motivate a further study of arithmetic and geometric series. We also look at amortization of loans to show additional applications of geometric series.

### Units 7-13 Resources

page 531	<b>Skill Builders (Extra Practice)</b> These problems provide concentrated sample problems and additional practice to the basic skills developed in the second half of the course.
page 537	<b>Extra Problems</b> These problems are a random mixture of additional practice that can supplement the current units or act as a review for final exams.
page 555	<b>Appendix B: Yearbook Sales: Matrices</b> The very important topic of matrices is introduced in real contexts and the motivation for the definitions of adding and multiplying matrices is developed. We see how matrices can be used to solve systems of equations. We utilize the matrix functions of graphing calculators to solve the more complex systems.
page 585	<b>Appendix C: Induction, Powers and Roots of Complex Numbers (De Moivre's Theorem), and Fundamental Theorem of Algebra</b>
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