

The students revisit circles in the first part of Chapter 10 to develop “circle tools,” which will help them find lengths and angle measures within circles. In addition to working with the length of the radius and diameter of a circle, they will add information about angles, arcs, and chords. As with many topics we have studied, this development will utilize triangles.

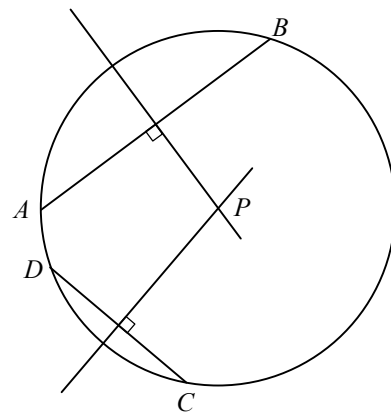
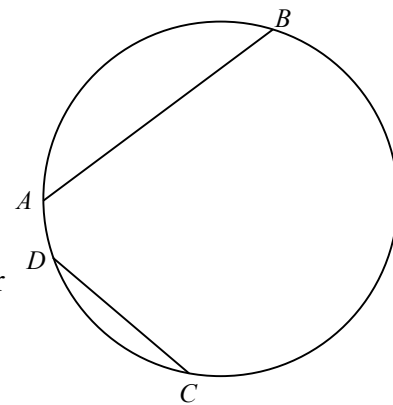
See the Math Notes boxes on pages 485, 490, 494, 498, and 501.

**Example 1**

In the circle at right are two chords,  $\overline{AB}$  and  $\overline{CD}$ . Find the center of the circle and label it  $P$ .

The chords of a circle (segments with endpoints on the circle) are useful segments. In particular, the diameter is a special chord that passes through the center. The perpendicular bisectors of the chords pass through the center of the circle as well. Therefore to find the center, we will find the perpendicular bisectors of each segment. They will meet at the center.

There are several ways to find the perpendicular bisectors of the segments. A quick way is to fold the paper so that the endpoints of the chords come together. The crease will be perpendicular to the chord and bisect it. Another method is to use the construction we learned last chapter. In either case, the point  $P$  is the center of the circle.



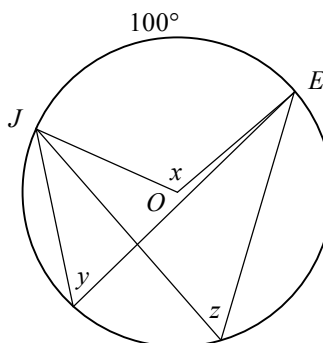
## Example 2

In  $\odot O$  at right, use the given information to find the values of  $x$ ,  $y$ , and  $z$ .

Pieces of a circle are called arcs, and every arc breaks the circle into two pieces. The large piece is called a major arc, and the smaller piece is called a minor arc. Arcs have lengths, and we found lengths of arcs by finding a fraction of the circumference. But arcs also have measures based on the measure of the central angle. In the picture above,  $\angle JOE$  is a central angle since its vertex is at the center,  $O$ .

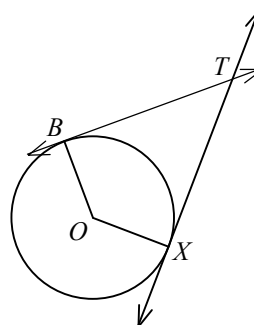
An arc's measure is the same as its central angle. Since  $\widehat{JE} = 100^\circ$ ,  $x = 100^\circ$ .

An angle with its vertex on the circle is called an inscribed angle. Both of the angles  $y$  and  $z$  are inscribed angle. Inscribed angles measure half of their intercepted arc (in this case,  $\widehat{JE}$ ). Therefore,  $y = z = \frac{1}{2}(100^\circ) = 50^\circ$ .

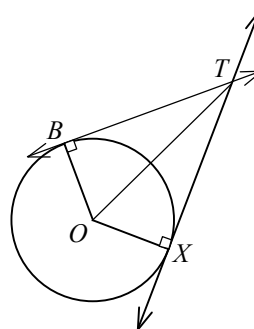


## Example 3

In the figure at right,  $O$  is the center of the circle.  $\overline{TX}$  and  $\overline{TB}$  are tangent to  $\odot O$ , and  $m\angle BOX = 120^\circ$ . Find the  $m\angle BTX$ .



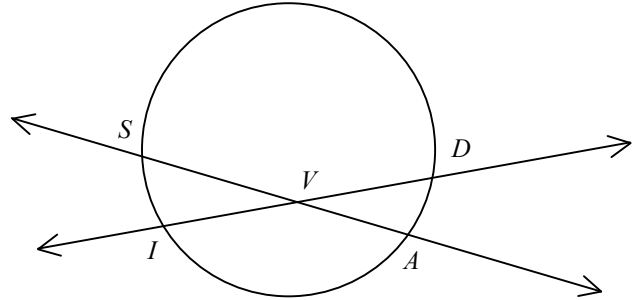
If a line is tangent to a circle, that line intersects the circle in only one point. Also, a radius drawn to the point of tangency is perpendicular to the tangent line. Therefore we know that  $\overline{OB} \perp \overline{BT}$  and  $\overline{OX} \perp \overline{XT}$ . At this point there are different ways to solve this problem. One way is to add a segment to the picture. Adding  $\overline{OT}$  will create two triangles, and we know a lot of information about triangles. In fact, these two right triangles are congruent by HL  $\cong$  ( $\overline{OB} \cong \overline{OX}$  because they are both radii, and  $\overline{OT} \cong \overline{OT}$ ). Since the corresponding parts of congruent triangles are also congruent, and  $m\angle BOX = 120^\circ$ , we know that  $m\angle BOT = m\angle XOT = 60^\circ$ . Using the sum of the angles of a triangle is  $180^\circ$ , we find  $m\angle BTO = m\angle XTO = 30^\circ$ . Therefore,  $m\angle BTX = 60^\circ$ .



An alternate solution is to note that the two right angles at points  $B$  and  $X$ , added to  $\angle BOX$ , make  $300^\circ$ . Since we know that the angles in a quadrilateral sum to  $360^\circ$ ,  $m\angle BTX = 360^\circ - 300^\circ = 60^\circ$ .

### Example 4

In the circle at right,  $DV = 9$  units,  $SV = 12$  units, and  $AV = 4$  units. Find the length of  $IV$ .



Although we have been concentrating on angles and their measures, there are some facts about lengths of chords of circles that are useful (and should be part of your “circle tools”). In the figure above, if we drew  $\overline{SI}$  and  $\overline{DA}$  we would form two similar triangles. (See the Math Notes box on page 498.) The sides of similar triangles are proportional, so we can write the proportion at right, which leads to the simplified equation with the two products.

$$\frac{SV}{DV} = \frac{IV}{AV}$$

$$SV \cdot AV = DV \cdot IV$$

$$\frac{SV}{DV} = \frac{IV}{AV}$$

$$\frac{12}{9} = \frac{IV}{4}$$

$$9IV = 48$$

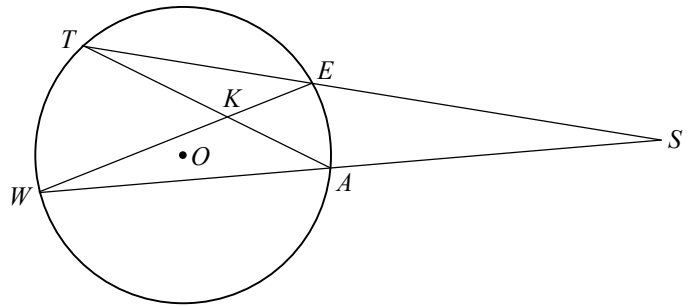
$$IV \approx 5.33 \text{ units}$$

Substitute the lengths that we know, then solve the equation.

### Problems

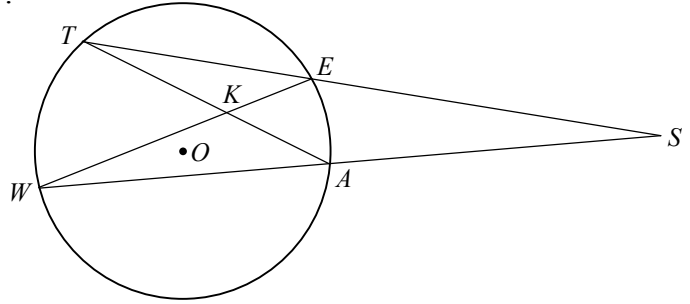
In  $\odot O$ ,  $m\widehat{WT} = 86^\circ$  and  $m\widehat{EA} = 62^\circ$ .

1. Find  $m\angle EWA$ .
2. Find  $m\angle WET$ .
3. Find  $m\angle WES$ .
4. Find  $m\angle WST$ .

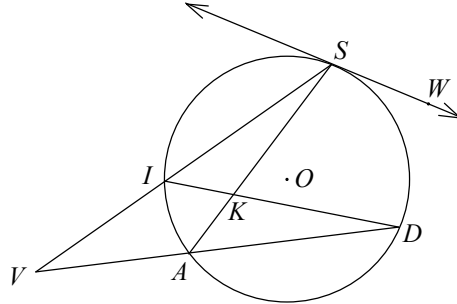


In  $\odot O$ ,  $m\angle EWA = 36^\circ$  and  $m\angle WST = 42^\circ$ .

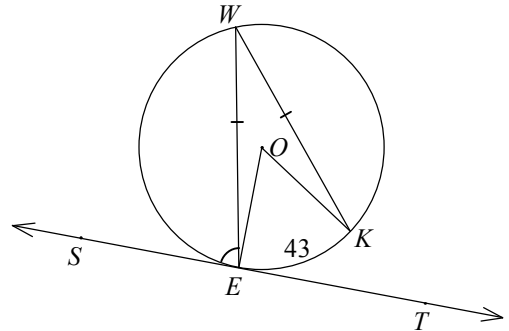
5. Find  $m\angle WES$ .
6. Find  $m\widehat{TW}$ .
7. Find  $m\widehat{EA}$ .
8. Find  $m\angle TKE$ .



9. In the figure at right,  $m\widehat{SD} = 92^\circ$ ,  $m\widehat{DA} = 103^\circ$ ,  $m\widehat{AI} = 41^\circ$  and  $\overline{SW}$  is tangent to  $\odot O$ . Find  $m\angle AKD$  and  $m\angle VAS$ .



10. In the figure at right,  $m\widehat{EK} = 43^\circ$ ,  $\overline{EW} \cong \overline{KW}$ , and  $\overline{ST}$  is tangent to  $\odot O$ . Find  $m\angle WEO$  and  $m\angle SEW$ .



### Answers

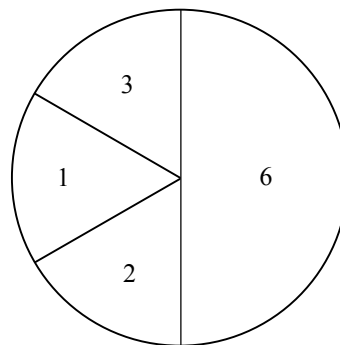
1.  $\frac{1}{2}(62^\circ) = 31^\circ$
2.  $\frac{1}{2}(86^\circ) = 43^\circ$
3.  $180^\circ - 43^\circ = 137^\circ$
4.  $180^\circ - 137^\circ - 31^\circ = 12^\circ$
5.  $180^\circ - 36^\circ - 42^\circ = 102^\circ$
6.  $m\angle TEW = 180^\circ - 102^\circ = 78^\circ$ ,  $2(78^\circ) = 156^\circ$
7.  $2(36^\circ) = 72^\circ$
8.  $180^\circ - 36^\circ - 78^\circ = 66^\circ$
9.  $m\angle SAD = \frac{1}{2}(92^\circ)$ ,  $m\angle IDA = \frac{1}{2}(41^\circ)$ ,  $180^\circ - 46^\circ - 20.5^\circ = 113.5^\circ$ ,  
 $m\angle VAS = 180^\circ - 46^\circ = 134^\circ$
10.  $m\angle EWK = \frac{1}{2}(43^\circ) = 21.5^\circ$ ,  $m\angle EOK = 43^\circ$ , so  $317^\circ$  remain for the other angle at  $O$ .  
 $m\angle WEO = m\angle WKO$  and for  $WEOK$ ,  
 $360^\circ - 21.5^\circ - 317^\circ = 21.5^\circ = m\angle WEO + m\angle WKO$ , so  
 $m\angle WEO = \frac{1}{2}(21.5^\circ) = 10.75^\circ$ .  $m\angle SEO = 90^\circ$ ,  $m\angle WEO = 10.75^\circ$ , so  
 $m\angle SEW = 79.25^\circ$ .

The students next investigate probability and expected value. Utilizing what the students already know about probability, their knowledge is extended, first through collecting data, and then by developing the mathematics behind the problem. Ultimately, the students come up with a formula for calculating the expected value for each play of a game. Something to be careful of in this section: students sometimes think that the expected value must actually be one of the possible outcomes. It does not have to be. The expected value tells what is the average expected result for one play.

See the Math Notes boxes on pages 505 and 516.

**Example 1**

The spinner at right is divided into different sections, each assigned a different point value. The three smaller sections are congruent. If you were to spin the spinner 100 times, how many times would you expect to get each of the different point values? What is the expected value of this spinner?



The size of each region is what determines the probability of the spinner landing in that region. Therefore the probability of landing on 6 points is  $\frac{1}{2}$  because that region takes up half of the spinner. The other half of the circle is divided into three equal parts, each taking up  $\frac{1}{6}$  of the whole spinner ( $\frac{1}{3}$  of  $\frac{1}{2}$ ). Now that we know the probabilities, we can determine how many times we would expect the values to come up. Since the probability of getting 6 points is  $\frac{1}{2}$ , we would expect that about half of the 100 spins would land on the 6, so 50 times. Similarly, since the probability of landing on 1 point (or 2 or 3) is  $\frac{1}{6}$ , would we expect about  $\frac{1}{6}$  of the 100 spins to land on each of those, or about 16 or 17 times. If the total number of spins is 100, we can expect on average about 50 of them to be 6 points,  $16\frac{2}{3}$  to be 1 point,  $16\frac{2}{3}$  to be 2 points, and  $16\frac{2}{3}$  to be 3 points. (Note: these are estimates, not exact or guaranteed.) Using these numbers, after 100 spins, the player would have about  $50(6) + 16\frac{2}{3}(1) + 16\frac{2}{3}(2) + 16\frac{2}{3}(3) = 400$  points. If the player earns 400 points in 100 spins, then on average the player received 4 points per spin. So for any single spin, the expected value is four points. Note: four is the expected value for this spinner, but it is NOT one of the possible outcomes.

## Example 2

A  $3 \times 3$  grid of nine congruent squares, each with a side length of 2 inches, is painted various colors. Six of the small squares are painted red while three are painted blue. For \$1.00 a player can throw a dart at the grid. If the player hits a blue square, he is handed \$2.00. Is this a fair game? Justify your answer.

The definition of a “fair” game is one in which the expected value is 0 because this means that, on average, the player is not guaranteed to win, and neither is the person running the game. If the expected value is 0, then winning or losing is just a matter of luck, and the game does not favor one side over the other. To determine if this game is fair we need to calculate its expected value.

Although we could go through a procedure similar to what we did in the last problem, there is a formula that is derived from that procedure that we can use. The expected value is found by summing the products of the amount won and its probability. In this problem, each game costs \$1.00.

If the dart lands on a red square, the player loses \$1.00 (the value is -1). The probability of hitting a red square is  $\frac{6}{9} = \frac{2}{3}$ .

However, if the player hits a blue square, the player receives \$2.00, which wins only \$1.00 (because he paid \$1.00 for the dart). Based on the calculations at right, the expected value

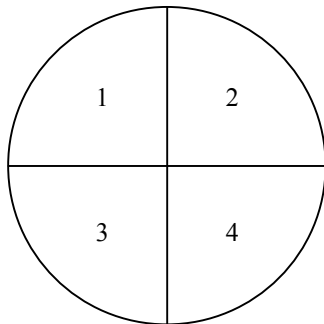
is  $-\frac{1}{3}$ . This tells us that on average the player can expect to lose  $\frac{1}{3}$  of a dollar, or about \$0.33, each turn. Therefore, this is not a fair game; it favors the person running the game (often known as “the house”).

$$\begin{aligned} E &= \frac{2}{3}(-1.00) + \frac{1}{3}(1.00) \\ &= -\frac{2}{3} + \frac{1}{3} \\ &= -\frac{1}{3} \end{aligned}$$

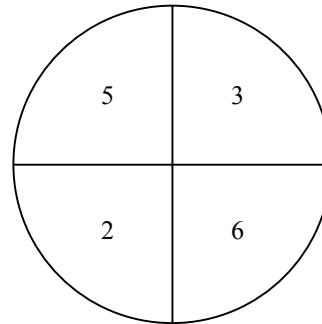
## Problems

The spinners below have different point values assigned to the different regions. What is the expected value for each spinner? (Assume that regions that appear to be congruent, are congruent.)

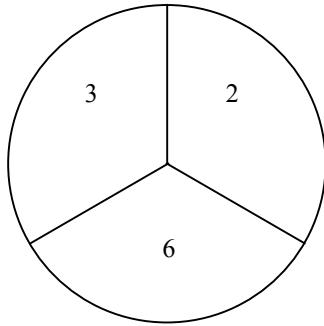
1.



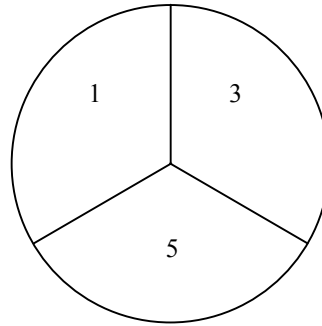
2.



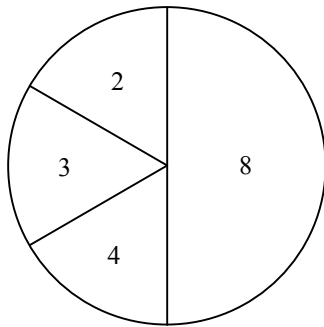
3.



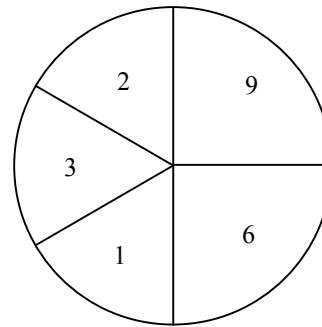
4.



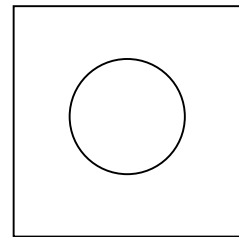
5.



6.



7. For \$2.00 a player gets five darts to throw at a board that looks like the figure at right. The board is a square, measuring one foot along each side. The circle is centered and has a diameter of six inches. For each dart that lands in the interior of the circle, the player gets \$0.75. Is this game fair? Justify your answer.



### Answers

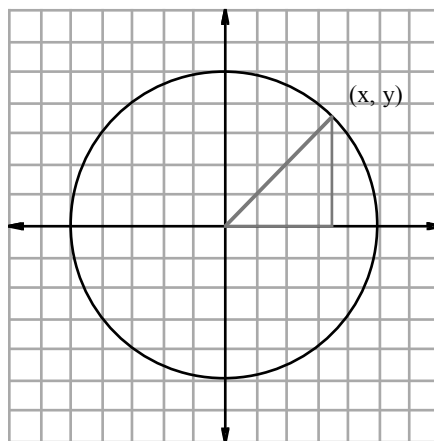
1. 2.5                      2. 4                      3.  $\approx 3.7$   
 4. 3                        5. 5.5                    6. 4.75  
 7. Not fair because the expected value is about  $-\$1.25$ .

The students have examined the parts of circles, found measurements of their chords and angles, and have used them with probability and expected value. This section places the circle on the coordinate grid so that the students can derive the equation of a circle.

**Example 1**

What is the equation of the circle centered at the origin with a radius of 5 units?

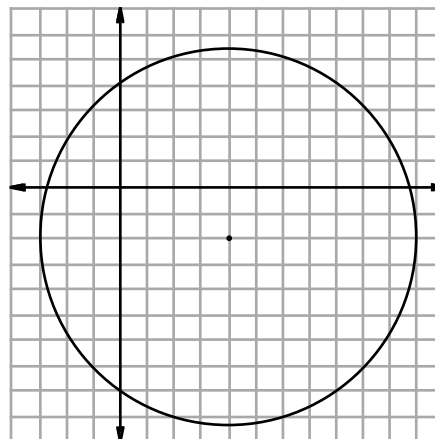
The key to deriving the equation of this circle is the Pythagorean Theorem. That means we will need to create a right triangle within the circle. First, draw the circle on graph paper, then choose any point on the circle. We do not know the exact coordinates of this point so call it  $(x, y)$ . Since endpoints of the radius are  $(0,0)$  and  $(x, y)$ , we can represent the length of the vertical leg as  $y$  and the length of the horizontal leg as  $x$ . If we call the radius  $r$ , then using the Pythagorean Theorem we can write  $x^2 + y^2 = r^2$ . Since we know the radius is 5, we can write the equation of this circle as  $x^2 + y^2 = 5^2$ , or  $x^2 + y^2 = 25$ .



**Example 2**

Graph the circle  $(x - 4)^2 + (y + 2)^2 = 49$ .

Based on what we have seen, this is a circle with a radius of 7. This one, however, is not centered at the origin. The general equation of a circle is  $(x - h)^2 + (y - k)^2 = r^2$ . The center of the circle is represented by  $(h, k)$ , so in this example the center is  $(4, -2)$ . The center of this circle has been shifted four units to the right, and two units down.



## Problems

1. What is the equation of the circle centered at  $(0, 0)$  with a radius of 25?
2. What is the equation of the circle centered at the origin with a radius of 7.5?
3. What is the equation of the circle centered at  $(5, -3)$  with a radius of 9?

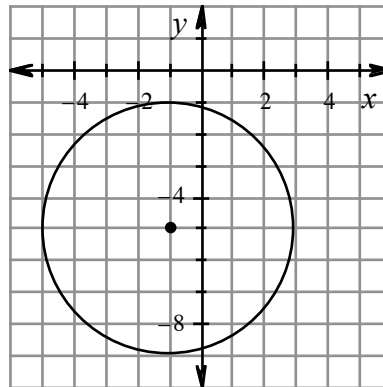
Graph the following circles.

4.  $(x+1)^2 + (y+5)^2 = 16$
5.  $x^2 + (y-6)^2 = 36$
6.  $(x-3)^2 + y^2 = 64$

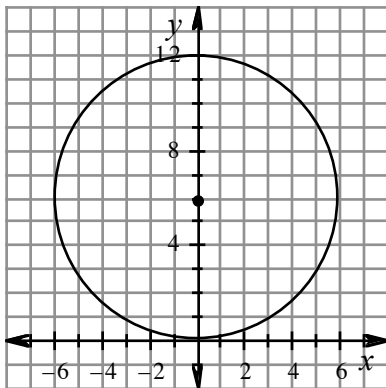
## Answers

1.  $x^2 + y^2 = 625$
2.  $x^2 + y^2 = 56.25$
3.  $(x-5)^2 + (y+3)^2 = 81$

4.



5.



6.

