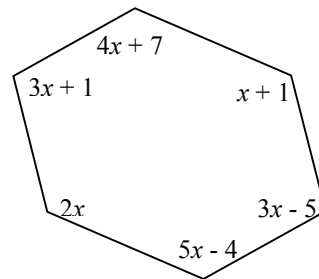


After studying triangles and quadrilaterals, the students now extend their knowledge to all polygons. A polygon is a closed, two-dimensional figure made of three or more non-intersecting straight line segments connected end-to-end. Just as the students were able to determine that the sum of the measures of the angles of a triangle is 180° , the students learn a method to determine the sum of the measures of the interior angles of any polygon. Next they explore the sum of the exterior angles of a polygon. Finally they use the information about the angles of polygons along with their Triangle Toolkit to find the areas of regular polygons.

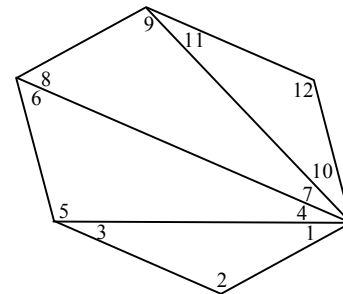
See the Math Notes boxes on pages 393, 395, 396, 400, 407, 410, and 422.

Example 1

The figure at right is a hexagon. What is the sum of the measures of the interior angles of a hexagon? Explain how you know. Then write an equation and solve for x .



One way to find the sum of the interior angles of the hexagon is to divide the figure into triangles. There are several different ways to do this, but keep in mind that we are trying to add the interior angles at the vertices, so those are the angles we want to see. One easy way to divide the hexagon is to draw in all the diagonals from one vertex as shown at right. Doing this forms four triangles, each with angle measures summing to 180° .



$$\underbrace{m\angle 1 + m\angle 2 + m\angle 3}_{180^\circ} + \underbrace{m\angle 4 + m\angle 5 + m\angle 6}_{180^\circ} + \underbrace{m\angle 7 + m\angle 8 + m\angle 9}_{180^\circ} + \underbrace{m\angle 10 + m\angle 11 + m\angle 12}_{180^\circ} = 4(180^\circ) = 720^\circ$$

(Note: students may have noticed that the number of triangles is always two less than the number of sides. This example illustrates why the sum of the interior angles of a polygon may be calculated using $(n - 2)180^\circ$, where n is the number of sides of the polygon.)

Now that we know what the sum of the angles is, we can write an equation, and solve for x .

$$\begin{aligned} (3x + 1) + (4x + 7) + (x + 1) + (3x - 5) + (5x - 4) + (2x) &= 720^\circ \\ 18x &= 720^\circ \\ x &= 40^\circ \end{aligned}$$

Example 2

If the sum of the measures of the angles of a polygon is 2340° , how many sides does the polygon have?

Use the equation “sum of interior angles = $(n - 2)180^\circ$ ” to write an equation and solve for n . The last line gives us an equation that we can solve.

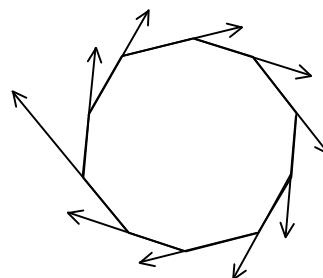
Therefore, the polygon has 15 sides. It is important to note that if the answer is not a whole number, something went wrong. Either we made a mistake or there is no polygon with its interior angles summing to the amount given. Since the answer is the number of sides, the answer can only be a whole number. Polygons cannot have “7.2” sides!

$$\begin{aligned}(n - 2) \cdot 180^\circ &= 2340^\circ \\ 180n - 360 &= 2340 \\ 180n &= 2700 \\ n &= 15\end{aligned}$$

Example 3

What is the measure of an exterior angle of a regular decagon?

A decagon is a 10-sided polygon. Since this figure is a regular decagon, all the angles and all the sides are congruent. The sum of the measures of the exterior angles of any polygon, one at each vertex, is always 360° , no matter how many sides the polygon has. The exterior angles are congruent since the decagon is regular. The decagon at right has ten exterior angles drawn, one at each vertex. Therefore, each angle measures $\frac{360^\circ}{10} = 36^\circ$.



Example 4

A regular dodecagon (12 sided polygon) has a side length of 8 units. What is its area?

Solving this problem is going to require the use of several topics we have studied. (Note: there is more than one way to solve this problem.) For this solution, we will imagine dividing the dodecagon into 12 triangles, radiating from the center. These triangles are all congruent to each other. If we could find the area of one of them, then we could multiply it by 12 to get the area of the entire figure.

To focus on one triangle, copy and enlarge it. The triangle is isosceles, so drawing a segment from the vertex angle perpendicular to the base gives a height. This height also bisects the base (because this triangle is isosceles).

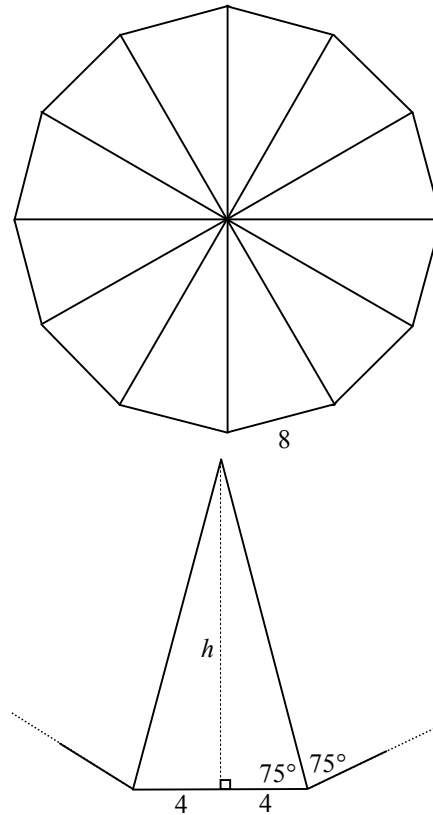
Since this is a dodecagon, we can find the sum of all the angles of the shape by using the formula the students developed:

$$(12 - 2)(180^\circ) = 1800^\circ$$

Since all the angles are congruent, each angle measures $1800^\circ \div 12 = 150^\circ$. The segments radiating from the center bisect each angle, so the base angle of the isosceles triangle is 75° . Now we can use trigonometry to find h .

Therefore the area of one of these triangles is

To find the area of the dodecagon, we multiply the area of one triangle by 12.



$$\begin{aligned}\tan 75^\circ &= \frac{h}{4} \\ h &= 4 \tan 75^\circ \\ h &\approx 14.928\end{aligned}$$

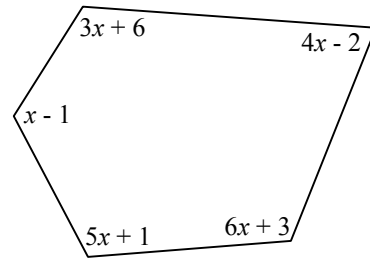
$$A \approx \frac{1}{2}(8)(14.928)$$

$$A \approx 59.712 \text{ square units}$$

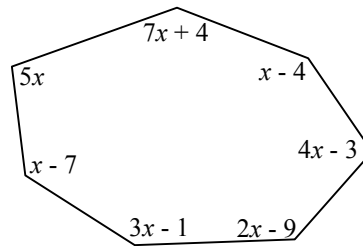
$$A \approx 12(59.712) \approx 716.544 \text{ square units}$$

Problems

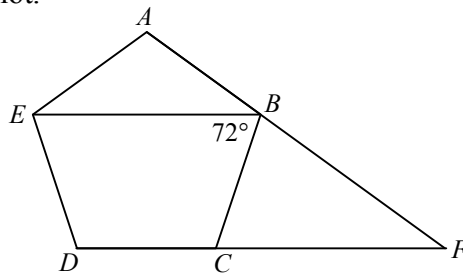
1. Using the pentagon at right, write an equation and solve for x .



2. Using the heptagon (7-gon) at right, write an equation and solve for x .



3. What is the sum of the measures of the interior angles of a 14-sided polygon?
4. What is the measure of each interior angle of a regular 16-sided polygon?
5. What is the sum of the measures of the exterior angles, one at each vertex, of a decagon (10-gon)?
6. Each exterior angle of a regular polygon measures 22.5° . How many sides does the polygon have?
7. Does a polygon exist whose sum of the interior angles is 3060° ? If so, how many sides does it have? If not, explain why not.
8. Does a polygon exist whose sum of the interior angles is 1350° ? If so, how many sides does it have? If not, explain why not.
9. Does a polygon exist whose sum of the interior angles is 4410° ? If so, how many sides does it have? If not, explain why not.
10. In the figure at right, $ABCDE$ is a regular pentagon. Is $\overline{EB} \parallel \overline{DF}$? Justify your answer.



11. What is the area of a regular pentagon with a side length of 10 units?
12. What is the area of a regular 15-gon with a side length of 5 units?

Answers

1. $19x + 7 = 540$,
 $x \approx 28.05$
2. $23x - 20 = 900$,
 $x = 40$
3. 2160°
4. 157.5°
5. 360°
6. 16 sides
7. 19 sides
8. No. The result is not a whole number. The number of sides must be a whole number.
9. No. The result is not a whole number.
10. Yes. Since $ABCDE$ is a regular pentagon, the measure of each interior angle is 108° . Therefore, $m\angle DCB = 108^\circ$. Since $\angle DCB$ and $\angle FCB$ are supplementary, $m\angle FCB = 72^\circ$. The lines are parallel because the alternate interior angles are congruent.
11. ≈ 172.0 sq. units
12. ≈ 441.1 sq. units

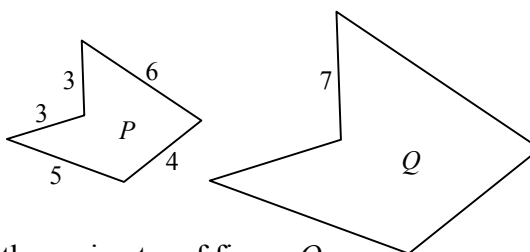
The students return to similarity once again to explore what happens to the area of a figure if it is reduced or enlarged. In Chapter 3, students discussed the ratio of similarity, also called the “zoom factor.” If two similar figures have a ratio of similarity of $\frac{a}{b}$, then the ratio of their perimeters is also $\frac{a}{b}$, while the ratio of their areas is $\frac{a^2}{b^2}$.

See the Math Notes boxes on pages 415 and 457.

Example 1

The figures P and Q at right are similar.

- a) What is the ratio of similarity?
- b) What is the perimeter of figure P ?
- c) Use your previous two answers to find the perimeter of figure Q .
- d) If the area of figure P is 34 square units, what is the area of figure Q ?



The ratio of similarity is the ratio of the lengths of two corresponding sides. In this case, since we only have the length of one side of figure Q , we will use the side of P that corresponds to that side. Therefore, the ratio of similarity is $\frac{3}{7}$.

To find the perimeter of figure P , add up all the side lengths: $3 + 6 + 4 + 5 + 3 = 21$. If the ratio of similarity of the two figures is $\frac{3}{7}$ then ratio of their perimeters is $\frac{3}{7}$ as well.

$$\begin{aligned} \frac{\text{perimeter } P}{\text{perimeter } Q} &= \frac{3}{7} \\ \frac{21}{Q} &= \frac{3}{7} \\ 3Q &= 147 \\ \text{perimeter } Q &= 49 \end{aligned}$$

If the ratio of similarity is $\frac{3}{7}$ then the ratio of the areas is $(\frac{3}{7})^2 = \frac{9}{49}$.

$$\begin{aligned} \frac{\text{area } P}{\text{area } Q} &= \left(\frac{3}{7}\right)^2 \\ \frac{34}{Q} &= \frac{9}{49} \\ 9Q &= 1666 \\ \text{area } Q &\approx 185.11 \text{ square units} \end{aligned}$$

Example 2

Two rectangles are similar. If the area of the first rectangle is 49 square units, and the area of the second rectangle is 256 square units, what is the ratio of similarity, $\frac{a}{b}$, between these two rectangles?

Since the rectangles are similar, if the ratio of similarity is $\frac{a}{b}$, $\frac{a^2}{b^2} = \frac{49}{256}$ then the ratio of their areas is $\frac{a^2}{b^2}$. We are given the areas so we know the ratio of their areas is $\frac{49}{256}$. Therefore we can write: $\frac{a}{b} = \sqrt{\frac{49}{256}} = \frac{\sqrt{49}}{\sqrt{256}} = \frac{7}{16}$

The ratio of similarity between the two rectangles is $\frac{a}{b} = \frac{7}{16}$. This can be written as a decimal or left as is.

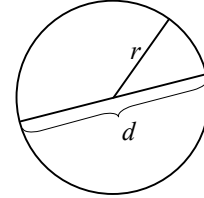
Problems

1. If figure A and figure B are similar with a ratio of similarity of $\frac{5}{4}$, and the perimeter of figure A is 18 units, what is the perimeter of figure B ?
2. If figure A and figure B are similar with a ratio of similarity of $\frac{1}{8}$, and the area of figure A is 13 square units, what is the area of figure B ?
3. If figure A and figure B are similar with a ratio of similarity of 6, that is, 6 to 1, and the perimeter of figure A is 54 units, what is the perimeter of figure B ?
4. If figure A and figure B are similar and the ratio of their perimeters is $\frac{17}{6}$, what is their ratio of similarity?
5. If figure A and figure B are similar and the ratio of their areas is $\frac{32}{9}$, what is their ratio of similarity?
6. If figure A and figure B are similar and the ratio of their perimeters is $\frac{23}{11}$, does that mean the perimeter of figure A is 23 units and the perimeter of figure B is 11 units? Explain.

Answers

1. 14.4 un. 2. 832 sq. un. 3. 9 un. 4. $\frac{17}{6}$ 5. $\frac{\sqrt{32}}{\sqrt{9}} \approx \frac{5.66}{3} \approx 1.89$
6. No, it just tells us the ratio. Figure A could have a perimeter of 46 units while figure B has a perimeter of 22 units.

The students have found the area and perimeter of several polygons, triangles, quadrilaterals, pentagons, hexagons, etc. Next they consider what happens to the area as more and more sides are added to a polygon. By exploring the area of a polygon with many sides, they learn that the limit of a polygon is a circle. They extend what they know about the perimeter and area of polygons to circles, and find the relationships for the circumference (C) and area (A) of circles.



$$C = \pi d \text{ or } 2\pi r, \quad A = \pi r^2$$

“ C ” is the circumference of the circle (a circle’s perimeter), “ d ” is the diameter, and “ r ” is the radius. “ π ,” which is in both formulas, is by definition the ratio $\frac{\text{circumference}}{\text{diameter}}$, and it is always a constant for any size circle.

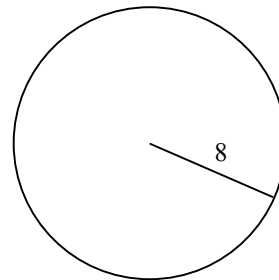
Using these formulas, along with ratios, the students are able to find the perimeter and area of shapes containing parts of circles.

See the Math Notes boxes on pages 426 and 430.

Example 1

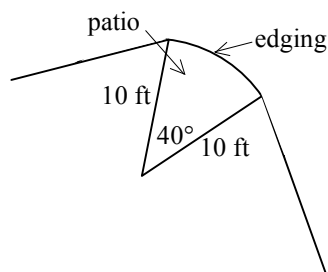
The circle at right has a radius of 8 cm. What are the circumference and the area of the circle? Using the formulas,

$$\begin{array}{ll} C = 2\pi r & A = \pi r^2 \\ = 2\pi(8) & = \pi(8)^2 \\ = 16\pi & = 64\pi \\ \approx 50.27 \text{ cm} & \approx 201.06 \text{ sq. cm} \end{array}$$



Example 2

Hermione has a small space on her corner lot that she would like to turn into a patio. To do this, she needs to do two things. First, she must know the length of the curved part, where she will put some decorative edging. Second, with the edging in place, she will need to purchase concrete to cover the patio. The concrete is sold in bags. Each bag will fill 2.5 square feet to the required depth of four inches. How much edging and concrete should Hermione buy?



The edging is a portion of the circumference of a circle with the center at point O and a radius of 10 feet. We can determine the exact fraction of the circle by looking at the measure of the central angle. Since the angle measures 40° , and there are 360° in the whole circle, this portion is $\frac{40^\circ}{360^\circ} = \frac{1}{9}$ of the circle. If we find the circumference and area of the whole circle, then we can take $\frac{1}{9}$ of each of those measurements to find the portion needed.

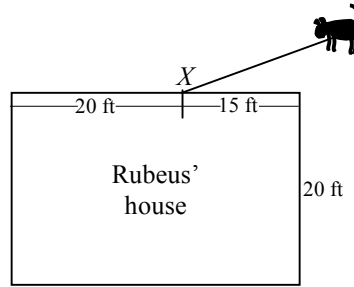
$$\begin{aligned} C &= \frac{1}{9}(2\pi r) \\ &= \frac{1}{9}(2 \cdot \pi \cdot 10) \\ &= \frac{20\pi}{9} \approx 6.98 \text{ feet} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{9}\pi r^2 \\ &= \frac{1}{9} \cdot \pi \cdot (10)^2 \\ &= \frac{100\pi}{9} \approx 34.91 \text{ square feet} \end{aligned}$$

Hermione should buy 7 feet of edging (most likely it is sold by the foot), and she should buy 14 bags of concrete ($34.91 \div 2.5 \approx 13.96$ bags). Concrete is sold in full bags only.

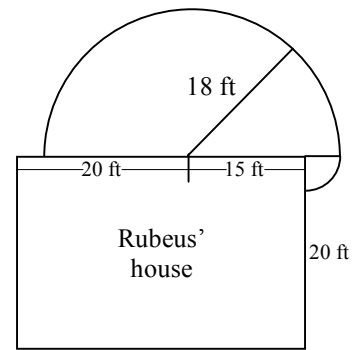
Example 3

Rubeus' dog Fluffy is tethered to the side of his house at point X . If Fluffy's rope is 18 feet long, how much area does Fluffy have to run in?



Because Fluffy is tethered to a point by a rope, he can only go where the rope can reach. Assuming that there are no obstacles, this area would be circular. Since Fluffy is blocked by the house, the area will only be a portion of a circle.

From point X , Fluffy can reach 18 feet to the left and right of point X . This initial piece is a semicircle. But, to the right of point X , the rope will bend around the corner of the house, adding a little more area for Fluffy. This smaller piece is a quarter of a circle with a radius of 3 feet.



Semicircle:

$$\begin{aligned} A &= \frac{1}{2} \pi r^2 \\ &= \frac{18^2 \pi}{2} \\ &= \frac{324 \pi}{2} \\ &= 162 \pi \approx 508.94 \end{aligned}$$

Quarter circle:

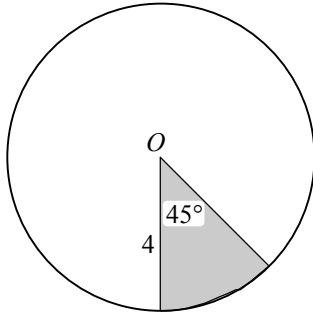
$$\begin{aligned} A &= \frac{1}{4} \pi r^2 \\ &= \frac{3^2 \pi}{4} \\ &= \frac{9 \pi}{4} \\ &\approx 7.07 \end{aligned}$$

Fluffy has a total of $508.94 + 7.07 \approx 516$ square feet in which to run.

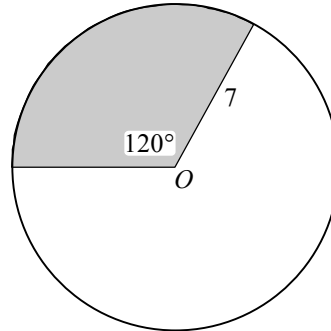
Problems

Find the area of the shaded sector in each circle below. In each case, point O is the center.

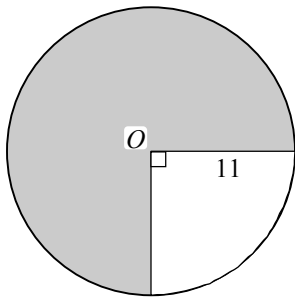
1.



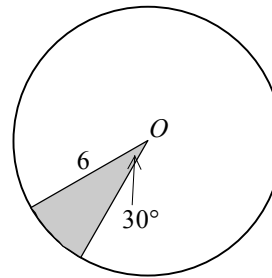
2.



3.

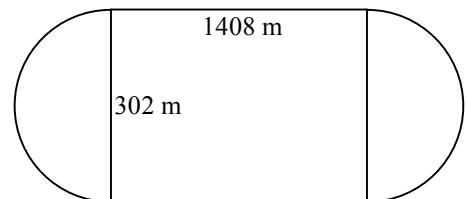


4.

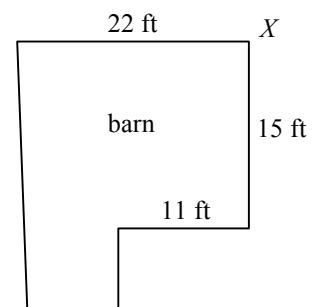


5. Find the perimeter of the shaded sector in problem 1.
6. Find the perimeter of the shaded sector in problem 2.
7. Find the perimeter of the shaded sector in problem 3.
8. Find the perimeter of the shaded sector in problem 4.

9. Kennedy and Tess are constructing a racetrack for their horses. The track encloses a field that is rectangular, with two semicircles at each end. A fence must surround this field. How much fencing will Kennedy and Tess need?



10. Rubens has moved his dog Fluffy to a corner of his barn because he wants him to have more room to run. If Fluffy is tethered at point X on the barn with a 20 foot rope, how much area does Fluffy have to explore?



Answers

1. $2\pi \approx 6.28$ square units
2. $\frac{49}{3}\pi \approx 51.31$ square units
3. $\frac{363\pi}{4} \approx 285.10$ square units
4. $3\pi \approx 9.42$ square units
5. $\pi + 8 \approx 11.14$ units
6. $\frac{14\pi}{3} + 14 \approx 28.66$ units
7. $\frac{33\pi}{2} + 22 \approx 73.84$ units
8. $\pi + 12 \approx 15.14$ units
9. $2816 + 302\pi \approx 3764.76$ meters of fencing
10. $200\pi + 100\pi + \frac{25\pi}{4} \approx 962.11$ square feet