

Measurement

and

Geometry

Area

Perimeter of Polygons and Circumference of Circles

Surface Area

Volume

The Pythagorean Theorem

Angles, Triangles, and Quadrilaterals

Scale Factor and Ratios of Growth

AREA

Area is the number of square units in the interior region of a plane (flat) figure (two dimensional) or the surface area of a three-dimensional figure. For example, area is the region that is covered by floor tile (two dimensional) or paint on a box or ball (three dimensional).

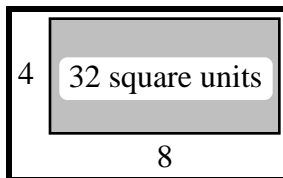
AREA OF A RECTANGLE

For the Tool Kit entry about area of a rectangle, see Year 1, Chapter 4, problem MP-16. To find the area of a rectangle, follow the steps below.

1. Identify the base.
2. Identify the height.
3. Multiply the base times the height to find the area in square units: $A = bh$.

A square is a rectangle in which the base and height are of equal length. Find the area of a square by multiplying the base times itself: $A = b^2$.

Example



base = 8 units

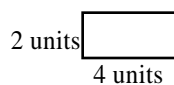
height = 4 units

$$A = 8 \cdot 4 = 32 \text{ square units}$$

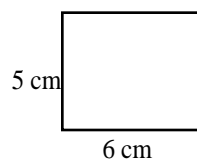
Problems

Find the areas of the rectangles and squares below.

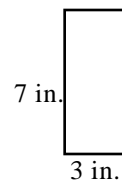
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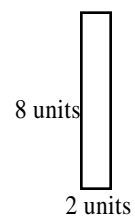
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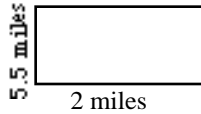
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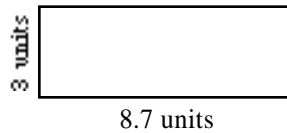
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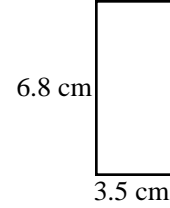
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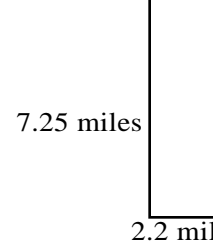
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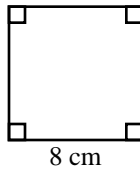
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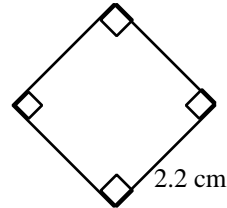
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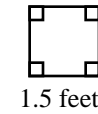
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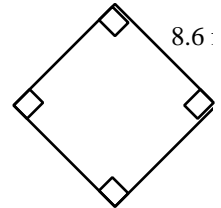
10.



11.



12.



Answers

1. 8 sq. units

2. 30 sq. cm

3. 21 sq. in.

4. 16 sq. units

5. 11 sq. miles

6. 26.1 sq. units

7. 23.8 sq. cm

8. 15.95 sq. miles

9. 64 sq. cm

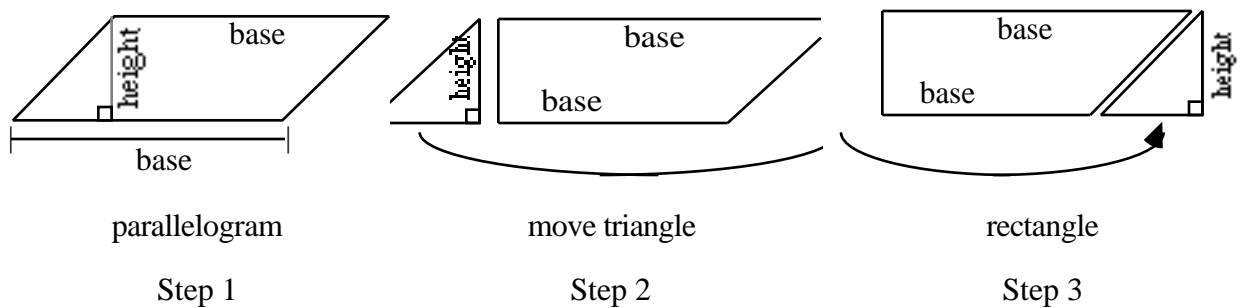
10. 4.84 sq. cm

11. 2.25 sq. feet

12. 73.96 sq. feet

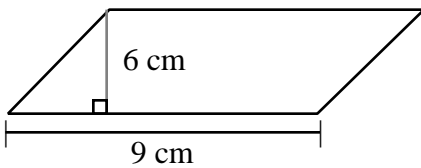
AREA OF A PARALLELOGRAM

A parallelogram is easily changed to a rectangle by separating a triangle from one end of the parallelogram and moving it to the other end as shown in the three figures below. For additional information, see Year 1, Chapter 4, problems MP-26 and MP-43 or Year 2, Chapter 6, problems RS-33 and 52.



To find the area of a parallelogram, multiply the base times the height as you did with the rectangle: $A = bh$.

Example



base = 9 cm

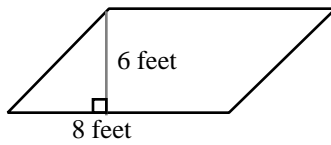
height = 6 cm

$$A = 9 \cdot 6 = 54 \text{ square cm}$$

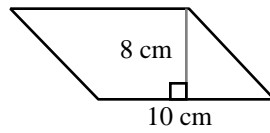
Problems

Find the area of each parallelogram below.

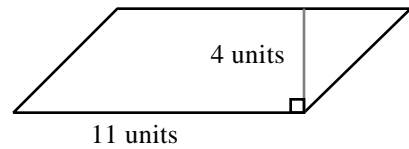
1.



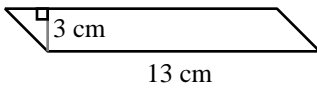
2.



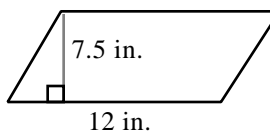
3.



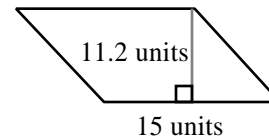
4.



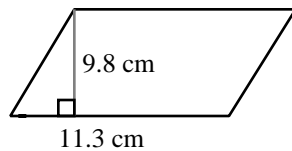
5.



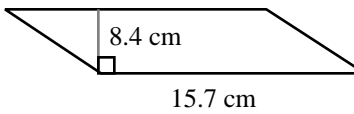
6.



7.



8.



Answers

1. 48 sq. feet

2. 80 sq. cm

3. 44 sq. units

4. 39 sq. cm

5. 90 sq. in.

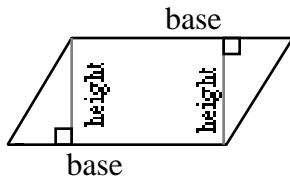
6. 168 sq. units

7. 110.74 sq. cm

8. 131.88 sq. cm

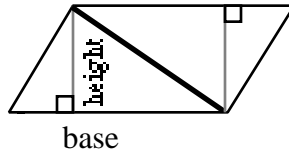
AREA OF A TRIANGLE

The area of a triangle is equal to one half the area of a parallelogram. This fact can easily be shown by cutting a parallelogram in half along a diagonal (see below). For additional information, see Year 1, Chapter 4, problem MP-55 or Year 2, Chapter 6, problem RS-52.



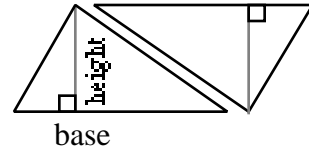
parallelogram

Step 1



draw a diagonal

Step 2



match triangles by cutting apart or by folding

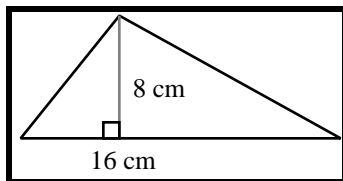
Step 3

As you match the triangles by either cutting the parallelogram apart or by folding along the diagonal, the result is two congruent (same size and shape) triangles. Thus, the area of a triangle has half the area of the parallelogram that can be created from two copies of the triangle.

To find the area of a triangle, follow the steps below.

1. Identify the base.
2. Identify the height.
3. Multiply the base times the height.
4. Divide the product of the base times the height by 2: $A = \frac{bh}{2}$ or $\frac{1}{2} bh$.

Example 1

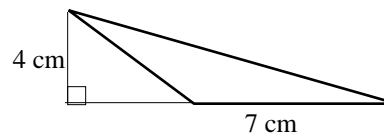


base = 16 cm

height = 8 cm

$$A = \frac{16 \cdot 8}{2} = \frac{128}{2} = 64 \text{ cm}^2$$

Example 2



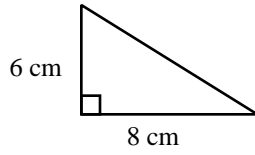
base = 7 cm

height = 4 cm

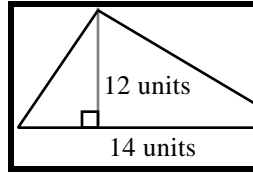
$$A = \frac{7 \cdot 4}{2} = \frac{28}{2} = 14 \text{ cm}^2$$

Problems

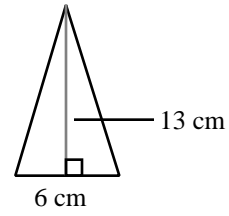
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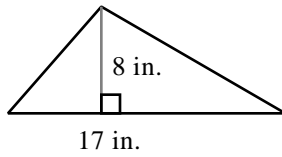
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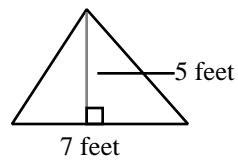
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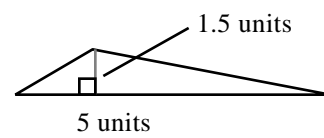
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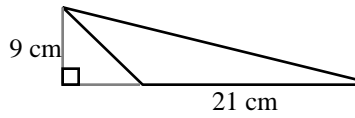
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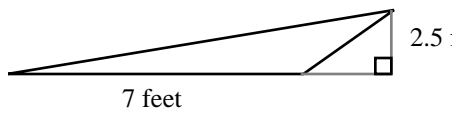
6.



7.



8.



Answers

1. 24 sq. cm

2. 84 sq. units

3. 39 sq. cm

4. 68 sq. cm

5. 17.5 sq. feet

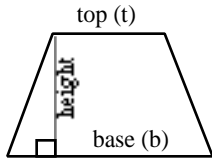
6. 3.75 sq. units

7. 94.5 sq. cm

8. 8.75 sq. feet

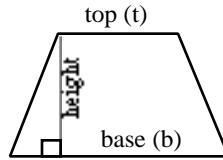
AREA OF A TRAPEZOID

A trapezoid is another shape that can be transformed into a parallelogram. Change a trapezoid into a parallelogram by following the three steps below.



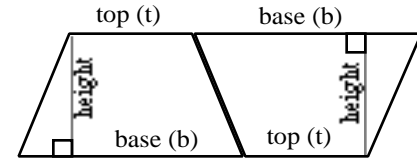
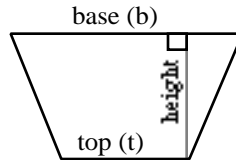
trapezoid

Step 1



duplicate the trapezoid and rotate

Step 2



put the two trapezoids together to form a parallelogram

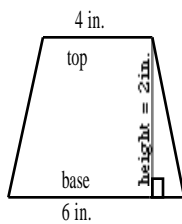
Step 3

To find the area of a trapezoid, multiply the base of the large parallelogram in step 3 (base and top) times the height and then take half of the total area. Remember to add the lengths of the base and the top of the trapezoid before multiplying by the height. Note that some texts call the top length the upper base and the base the lower base.

$$A = \frac{1}{2} (b + t) h \quad \text{or} \quad \frac{b + t}{2} \cdot h$$

For additional information, see Year 1, Chapter 4, problem MP-109 or Year 2, Chapter 6, problem RS-52.

Example



top = 4 in.

base = 6 in.

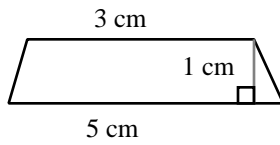
height = 2 in.

$$A = \frac{4 + 6}{2} \cdot 2 = \frac{10}{2} \cdot 2 = 5 \cdot 2 = 10 \text{ in.}^2$$

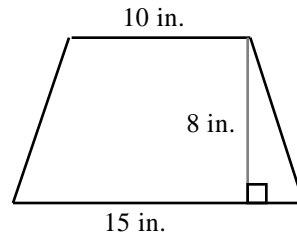
Problems

Find the areas of the trapezoids below.

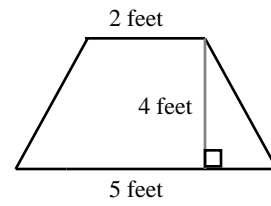
1.



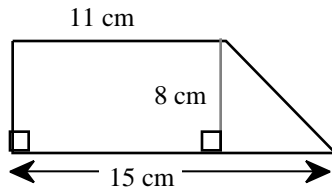
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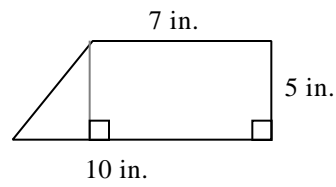
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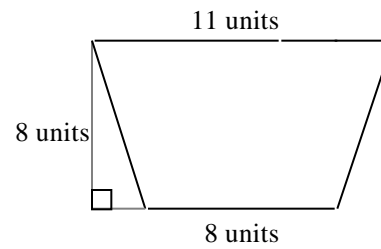
4.



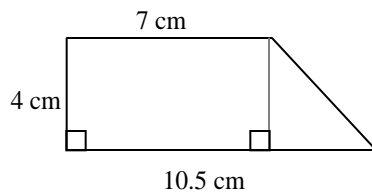
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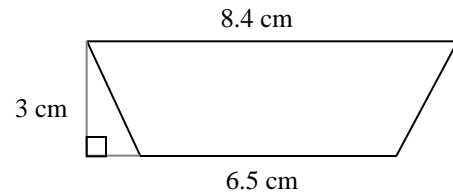
6.



7.



8.



Answers

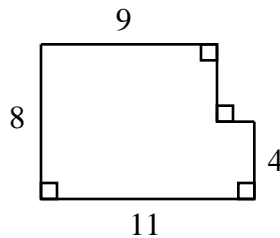
- | | | | | | | | |
|----|--------------|----|--------------|----|-------------|----|---------------|
| 1. | 4 sq. cm | 2. | 100 sq. in. | 3. | 14 sq. feet | 4. | 104 sq. cm |
| 5. | 42.5 sq. in. | 6. | 76 sq. units | 7. | 35 sq. cm | 8. | 22.35 sq. in. |

CALCULATING COMPLEX AREAS USING SUBPROBLEMS

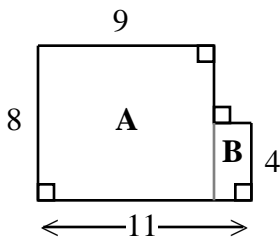
Students can use their knowledge of areas of polygons to find the areas of more complicated figures. The use of subproblems (that is, solving smaller problems in order to solve a larger problem) is one way to find the areas of complicated figures. For additional information, see Year 2, Chapter 8, problems GS-43 and 45.

Example 1

Find the area of the figure at right.



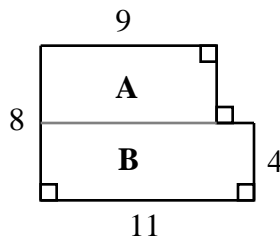
Method #1



Subproblems:

1. Find the area of rectangle A:
 $8 \cdot 9 = 72$ square units
2. Find the area of rectangle B:
 $4 \cdot (11 - 9) = 4 \cdot 2 = 8$ square units
3. Add the area of rectangle A to the area of rectangle B:
 $72 + 8 = 80$ square units

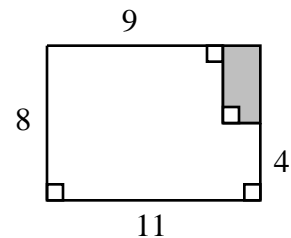
Method #2



Subproblems:

1. Find the area of rectangle A:
 $9 \cdot (8 - 4) = 9 \cdot 4 = 36$ square units
2. Find the area of rectangle B:
 $11 \cdot 4 = 44$ square units
3. Add the area of rectangle A to the area of rectangle B:
 $36 + 44 = 80$ square units

Method #3

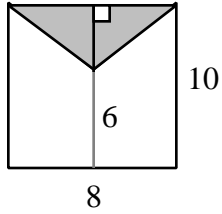
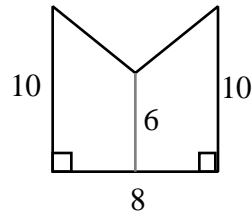


Subproblems:

1. Make a large rectangle by enclosing the upper right corner.
2. Find the area of the new, larger rectangle:
 $8 \cdot 11 = 88$ square units
3. Find the area of the shaded rectangle:
 $(8 - 4) \cdot (11 - 9)$
 $= 4 \cdot 2 = 8$ square units
4. Subtract the shaded rectangle from the larger rectangle:
 $88 - 8 = 80$ square units

Example 2

Find the area of the figure at right.



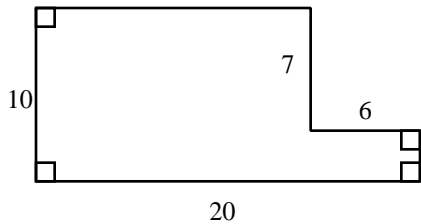
Subproblems:

1. Make a rectangle out of the figure by enclosing the top.
2. Find the area of the entire rectangle: $8 \cdot 10 = 80$ square units
3. Find the area of the shaded triangle. Use the formula $A = \frac{1}{2} bh$.
 $b = 8$ and $h = 10 - 6 = 4$, so $A = \frac{1}{2} (8 \cdot 4) = \frac{32}{2} = 16$ square units
4. Subtract the area of the triangle from the area of the rectangle:
 $80 - 16 = 64$ square units

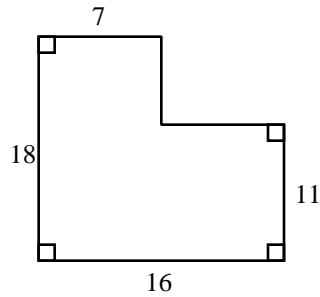
Problems

Find the areas of the complex figures below.

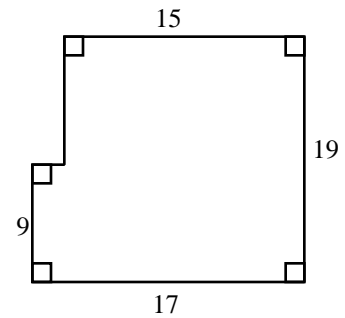
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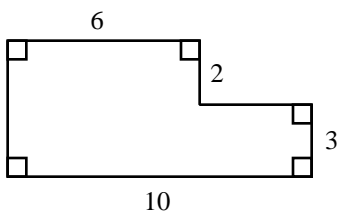
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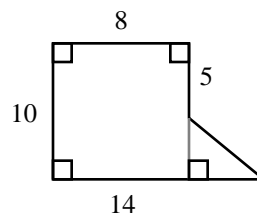
3.



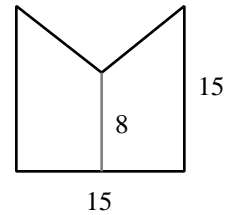
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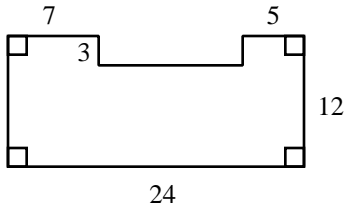
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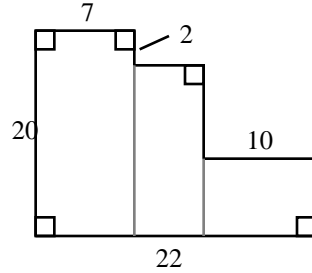
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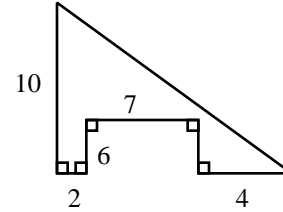
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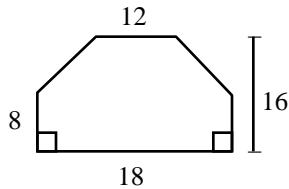
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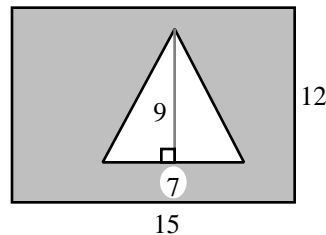
9.



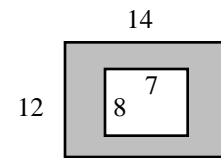
10.



11. Find the area of the shaded region.



12. Find the area of the shaded region.



Answers

- | | | | |
|------------------|--------------------|---------------------|-------------------|
| 1. 158 sq. units | 2. 225 sq. units | 3. 303 sq. units | 4. 42 sq. units |
| 5. 95 sq. units | 6. 172.5 sq. units | 7. 252 sq. units | 8. 310 sq. units |
| 9. 23 sq. units | 10. 264 sq. units | 11. 148.5 sq. units | 12. 112 sq. units |

AREA OF A CIRCLE

In class, students have done explorations with circles and circular objects to discover the relationship between circumference, diameter, and pi (π). To read more about the in-class exploration see Year 1, Chapter 9, problems ZC-10, 20, 21, 23, and 24 as well as the Tool Kit entry, Chapter 9, problem ZC-25. In Year 2, Chapter 6, see problem RS-61 as well as the Tool Kit entry, problem RS-63.

In order to find the area of a circle, students need to identify the radius of the circle. The radius is half the diameter. Next they will square the radius and multiply the result by π . Depending on the teacher's or book's preference, students may use $\frac{22}{7}$ for π when the radius or diameter is a fraction, 3.14 for π as an approximation, or the π button on the calculator. When using the π button, most teachers will want students to round to the nearest tenth or hundredth.

The formula for the area of a circle is: $A = \pi r^2$.

Example 1

Find the area of a circle with $r = 17$ feet.

$$\begin{aligned} A &= (17)^2 \\ &= 3.14(17 \cdot 17) \\ &= 907.36 \text{ square feet} \end{aligned}$$

Example 3

Find the radius of a circle with area 78.5 square meters.

$$\begin{aligned} 78.5 &= r^2 \\ 78.5 &= 3.14r^2 \\ 78.5 \div 3.14 &= 24.89 \\ \frac{78.5}{3.14} &= \frac{3.14}{3.14} r^2 \\ 24.88 &= r^2 \end{aligned}$$

$$r = \frac{\sqrt{24.88\dots}}{5} \text{ meters}$$

Example 2

Find the area of a circle with $d = 84$ cm.

$$\begin{aligned} r &= 42 \text{ cm} \\ A &= (42)^2 \\ &= 3.14(42 \cdot 42) \\ &= 5538.96 \text{ square cm} \end{aligned}$$

Example 4

Find the radius of a circle with area 50.24 square centimeters.

$$\begin{aligned} 50.24 &= r^2 \\ 50.24 &= 3.14r^2 \\ \frac{50.24}{3.14} &= \frac{3.14}{3.14} r^2 \\ 16 &= r^2 \end{aligned}$$

$$\begin{aligned} r &= \sqrt{16} \\ &= 4 \text{ centimeters} \end{aligned}$$

Problems

Find the area of circles with the following radius or diameter lengths. Round to the nearest hundredth.

- $r = 6$ cm
- $r = 3.2$ in.
- $d = 16$ ft
- $r = \frac{1}{2}$ m
- $d = \frac{4}{5}$ cm
- $r = 5$ in.
- $r = 3.6$ cm
- $r = 2\frac{1}{4}$ in.
- $d = 14.5$ ft
- $r = 12.02$ m

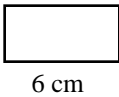
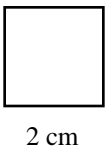
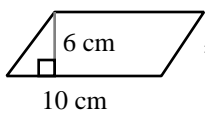
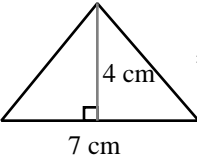
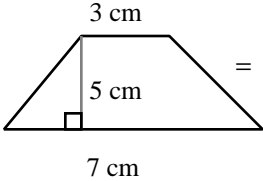
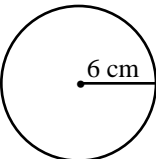
Find the radius of each circle given the following areas. Round answers to the nearest tenth.

- $A = 36.29$ m²
- $A = 63.59$ cm²
- $A = 153.86$ ft²
- $A = 530.66$ in²
- $A = 415.265$ km²

Answers

- 113.04 cm²
- 32.15 in²
- 200.96 ft²
- $\frac{11}{14}$ m²
- $\frac{88}{175}$ cm²
- 78.5 in²
- 40.69 cm²
- $15\frac{51}{56}$ or 15.90 in²
- 165.05 ft²
- 453.67 m²
- $r = 3.4$ m
- $r = 4.5$ cm
- $r = 7$ ft
- $r = 13$ in.
- $r = 11.5$ km

SUMMARY TABLE FOR AREA

Shape	Formula	Example
rectangle	$A = bh$	 $= 6 \cdot 3 = 18$ square cm
square	$A = b^2$	 $= 2 \cdot 2 = 4$ square cm
parallelogram	$A = bh$	 $= 10 \cdot 6 = 60$ square cm
triangle	$A = \frac{bh}{2}$ or $\frac{1}{2}bh$	 $= \frac{7 \cdot 4}{2} = \frac{28}{2} = 14$ square cm
trapezoid	$A = \frac{1}{2}(b + t)h$ or $\frac{b + t}{2} \cdot h$	 $= \frac{7 + 3}{2} \cdot 5 = 25$ square cm
circle	$A = r^2$	 $= 3.14(6)^2 = 113.04$ square cm

PERIMETER OF POLYGONS AND CIRCUMFERENCE OF CIRCLES

PERIMETER

The perimeter of a polygon is the distance around the outside of the figure. The perimeter is found by adding the lengths of all of the sides. The perimeter is similar to a fence put around a yard.

In the examples below, you may first need to find the length of the missing side(s) to find the perimeter.

Example 1

Find the perimeter of the parallelogram below.

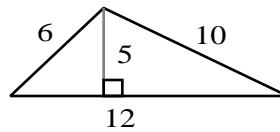


$$P = 6 + 4 + 6 + 4 = 20 \text{ units}$$

(Parallelograms have opposite sides equal.)

Example 2

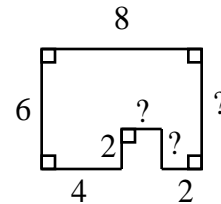
Find the perimeter of the triangle below.



$$P = 6 + 10 + 12 = 28 \text{ units}$$

Example 3

Find the perimeter of the figure below.



$$P = 6 + 8 + 6 + 2 + 2 + 2 + 2 + 4 = 32 \text{ units}$$

(You need to look carefully to find the lengths of the missing sides.)

Note: In some problems the height might be needed to do subproblems such as finding the length of the side of the figure. Since all of the sides are known in examples 1 and 2 above, the height is not used in the calculations.

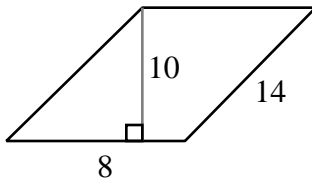
Problems

Find the perimeter of each shape.

1. a rectangle with $b = 5$ and $h = 10$

3. a parallelogram with $b = 8$ and $s = 5$

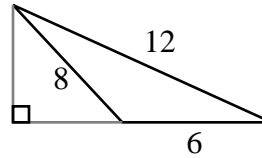
5. a parallelogram



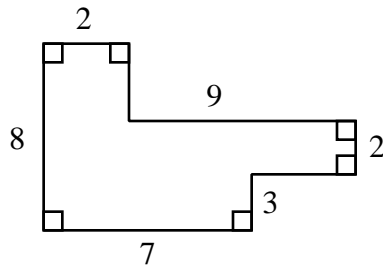
2. a square with sides of length 9

4. a triangle with sides 4, 10, and 12

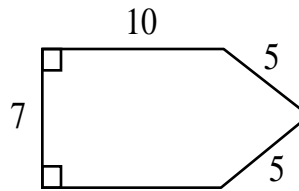
6.



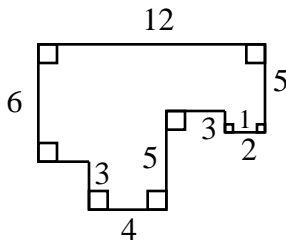
7.



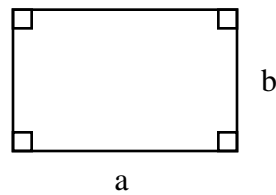
8.



9.



10.



Answers

1. 30 units

2. 36 units

3. 26 units

4. 26 units

5. 44 units

6. 26 units

7. 38 units

8. 37 units

9. 44 units

10. $a + b + a + b = 2a + 2b$

CIRCUMFERENCE

The circumference of a circle is similar to the perimeter of a polygon. The circumference is the length of a circle. The circumference would tell you how much string it would take to go around a circle once.

Circumference is explored by students in Year 1, Chapter 9, problem ZC-10 and Tool Kit entry ZC-11 and in Year 2, Chapter 6, problem RS-57 and Tool Kit entry RS-62. The ratio of the circumference to the diameter of a circle is pi (π). Circumference is found by multiplying π by the diameter. Students may use $\frac{22}{7}$, 3.14, or the π button on their calculator, depending on the teacher's or the book's directions.

$$C = 2 \pi r \text{ or } C = \pi d$$

For additional information, see Year 1, Chapter 9, problem ZC-11 or Year 2, Chapter 5, problem RS-62.

Example 1

Find the circumference of a circle with a diameter of 5 inches.

$$d = 5 \text{ inches}$$

$$\begin{aligned} C &= \pi d \\ &= \pi(5) \text{ or } 3.14(5) \\ &= 15.7 \text{ inches} \end{aligned}$$

Example 2

Find the circumference of a circle with a radius of 10 units.

$$r = 10, \text{ so } d = 2(10) = 20$$

$$\begin{aligned} C &= 3.14(20) \\ &= 62.8 \text{ units} \end{aligned}$$

Example 3

Find the diameter of a circle with a circumference of 163.28 inches.

$$\begin{aligned} C &= \pi d \\ 163.28 &= \pi d \\ 163.28 &= 3.14d \\ d &= \frac{163.28}{3.14} \\ &= 52 \text{ inches} \end{aligned}$$

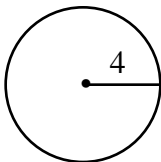
Problems

Find the circumference of each circle given the following radius or diameter lengths. Round your answer to the nearest hundredth.

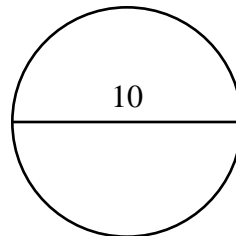
1. $d = 12$ 2. $d = 3.4$ 3. $r = 2.1$ 4. $d = 25$ 5. $r = 1.54$

Find the circumference of each circle shown below. Round your answer to the nearest hundredth.

6.



7.



Find the diameter of each circle given the circumference. Round your answer to the nearest tenth.

8. $C = 48.36$

9. $C = 35.6$

10. $C = 194.68$

Answers

1. 37.68 units

2. 10.68 units

3. 13.19 units

4. 78.5 units

5. 9.67 units

6. 25.12 units

7. 31.40 units

8. 15.4 units

9. 11.3 units

10. 62 units

SURFACE AREA

SURFACE AREA OF A PRISM

The surface area of a prism (SA) is the sum of the areas of all of the faces, including the bases. Surface area is expressed in square units.

For additional information, see Year 1, Chapter 9, problem ZC-84 and 87 or Year 2, Chapter 8 problems GS-90.

Example

Find the surface area of the triangular prism at right.

Subproblem 1: Area of the 2 bases

$$2\left[\frac{1}{2}(6\text{cm})(8\text{ cm})\right] = 48\text{ cm}^2$$

Subproblem 2: Area of the 3 sides (lateral faces)

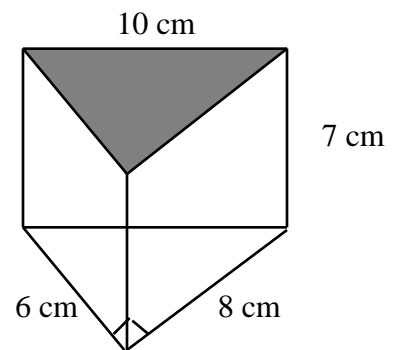
Area of side 1: $(6\text{ cm})(7\text{ cm}) = 42\text{ cm}^2$

Area of side 2: $(8\text{ cm})(7\text{ cm}) = 56\text{ cm}^2$

Area of side 3: $(10\text{ cm})(7\text{ cm}) = 70\text{ cm}^2$

Subproblem 3: Surface Area of Prism = sum of bases and lateral f

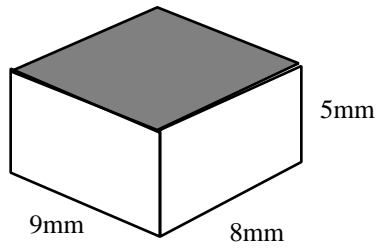
$$SA = 48\text{ cm}^2 + 42\text{ cm}^2 + 56\text{ cm}^2 + 70\text{ cm}^2 = 216\text{ cm}^2$$



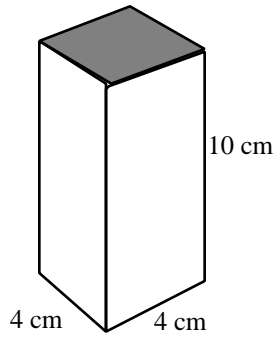
Problems

Find the surface area of each prism.

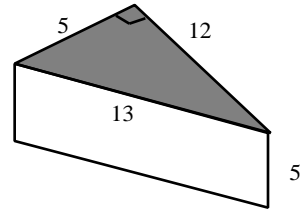
1.



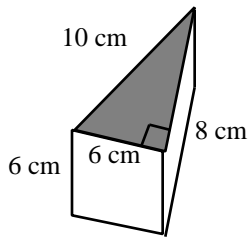
2.



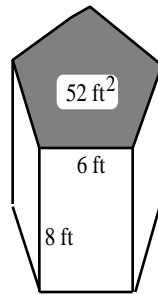
3.



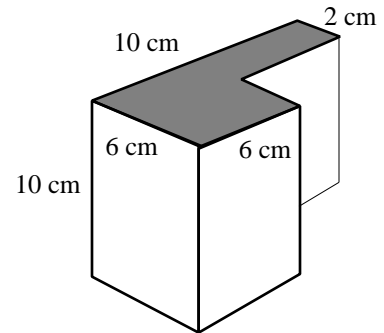
4.



5. The pentagon is equilateral. 6.



6.



Answers

1. 314 mm^2

2. 192 cm^2

3. 210 u^2

4. 192 cm^2

5. 344 ft^2

6. 408 cm^2

SURFACE AREA OF A CYLINDER

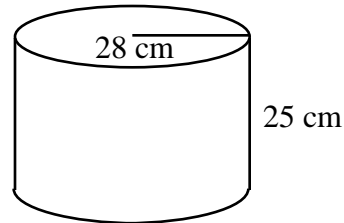
The surface area of a cylinder is the sum of the two base areas and the lateral surface area. The formula for the surface area is:

$$SA = 2 r^2 + dh \text{ or } SA = 2 r^2 + 2 rh$$

where r = radius, d = diameter, and h = height of the cylinder. For additional information, see Year 1, Chapter 9, problem ZC-87 or Year 2, Chapter 8, problem GS-123.

Example 1

Find the surface area of the cylinder at right.



Subproblem 1: Area of the two circular bases

$$2[(28 \text{ cm})^2] = 1568 \text{ cm}^2$$

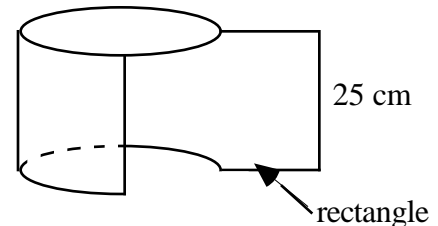
Subproblem 2: Area of the lateral face

$$(56)25 = 1400 \text{ cm}^2$$

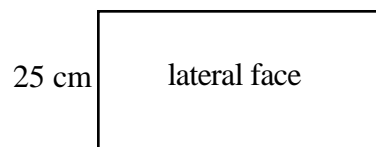
Subproblem 3: Surface area of the cylinder

$$1568 \text{ cm}^2 + 1400 \text{ cm}^2 = 2968 \text{ cm}^2$$

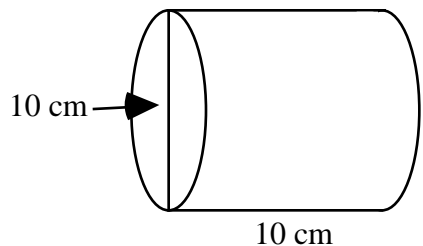
$$9324.25 \text{ cm}^2$$



circumference of base = 56 cm



Example 2



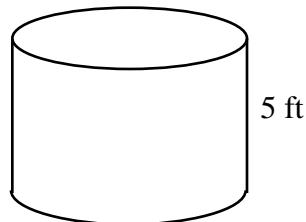
$$SA = 2 r^2 + 2 rh$$

$$= 2 (5)^2 + 2 \cdot 5 \cdot 10$$

$$= 50 + 100$$

$$= 150 \quad 471.24 \text{ cm}^2$$

Example 3



If the volume of the tank above is 1571.80 ft^3 , what is the surface area?

$$V = r^2 h$$

$$1571.80 = r^2(5)$$

$$\frac{1571.80}{5} = r^2$$

$$314.36 = r^2$$

$$17.73 = r$$

$$SA = 2 r^2 + 2 rh$$

$$= 2 (17.73)^2 + 2 (17.73)(5)$$

$$= 200 + 100$$

$$= 300 \quad 942.48 \text{ ft}^2$$

Problems

Find the surface area of each cylinder.

1. $r = 6$ cm, $h = 10$ cm
2. $r = 3.5$ in., $h = 25$ in.
3. $d = 9$ in., $h = 8.5$ in.
4. $d = 15$ cm, $h = 10$ cm
5. base area = 25, height = 8
6. Volume = 1000 cm³, height = 25 cm

Answers

1. 603.19 cm²
2. 626.75 in²
3. 367.57 in²
4. 824.69 cm²
5. 191.80 u²
6. 640.50 cm²

VOLUME

VOLUME OF A PRISM

Volume is a three-dimensional concept. It measures the amount of interior space of a three-dimensional figure based on a cubic unit, that is, the number of 1 by 1 by 1 cubes that will fit inside a figure. In this textbook series students will calculate the volume of prisms, cylinders, and cones.

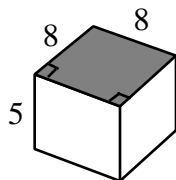
The volume of a prism is the area of either base (A) times the height (h) of the prism.

$$V = (\text{Area of base}) \cdot (\text{height}) \text{ or } V = Ah$$

For additional information, see Year 1, Chapter 9, problem ZC-58 or Year 2, Chapter 8, problem GS-79.

Example 1

Find the volume of the square prism below.

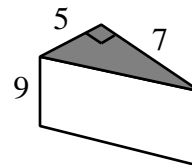


The base is a square with area (A)
 $8 \cdot 8 = 64$ units².

$$\begin{aligned} \text{Volume} &= A(h) \\ &= 64(5) \\ &= 320 \text{ units}^3 \end{aligned}$$

Example 2

Find the volume of the triangular prism below.

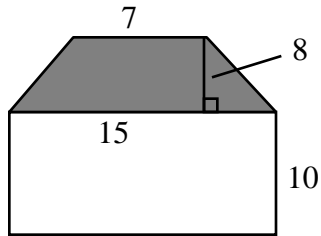


The base is a right triangle with area
 $\frac{1}{2}(5)(7) = 17.5$ units².

$$\begin{aligned} \text{Volume} &= A(h) \\ &= 17.5(9) \\ &= 157.5 \text{ units}^3 \end{aligned}$$

Example 3

Find the volume of the trapezoidal prism below.



The base is a trapezoid with area $\frac{1}{2}(7 + 15) \cdot 8 = 88 \text{ units}^2$.

$$\begin{aligned} \text{Volume} &= A(h) \\ &= 88(10) \\ &= 880 \text{ ur} \end{aligned}$$

Example 4

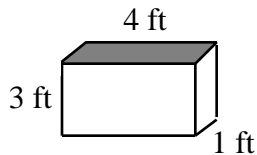
Find the height of the prism with a volume of 132.5 cm^3 and base area of 25 cm^2 .

$$\begin{aligned} \text{Volume} &= A(h) \\ 132.5 &= 25(h) \\ h &= \frac{132.5}{25} \\ h &= 5.3 \text{ cm} \end{aligned}$$

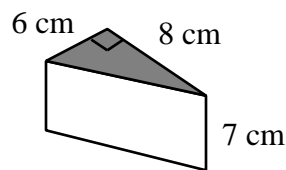
Problems

Calculate the volume of each prism. The base of each figure is shaded.

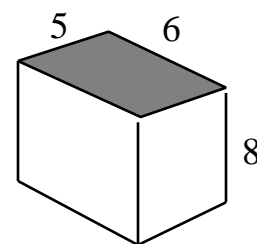
1. Rectangular Prism



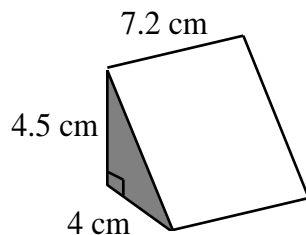
2. Right Triangular Prism



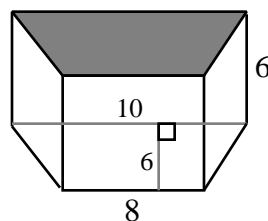
3. Rectangular Prism.



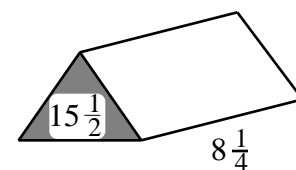
4. Right Triangular Prism



5. Trapezoidal Prism



6. Triangular Prism with $A = 15\frac{1}{2} \text{ units}^2$



7. Find the volume of a prism with base area 32 cm^2 and height 1.5 cm .
8. Find the height of a prism with base area 32 cm^2 and volume 176 cm^3 .
9. Find the base area of a prism with volume 47.01 cm^3 and height 3.2 cm .

Answers

1. 12 ft^3 2. 168 cm^3 3. 240 units^3 4. 64.8 cm^3 5. 324 units^3
6. $127\frac{7}{8} \text{ units}^3$ 7. 48 cm^3 8. 5.5 cm 9. 14.7 cm^2

VOLUME OF A CYLINDER

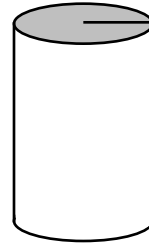
The volume of a cylinder is the area of its base multiplied by its height:

$$V = A \cdot h.$$

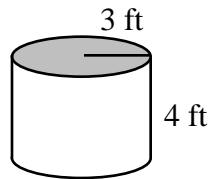
Since the base of a cylinder is a circle of area $A = r^2$, we can write:

$$V = r^2h.$$

For additional information, see Year 1, Chapter 9, problem ZC-72 or Year 2, Chapter 8, problem GS-108.



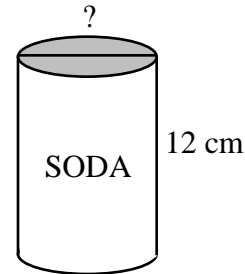
Example 1



Find the volume of the cylinder above.

$$\begin{aligned} \text{Volume} &= r^2h \\ &= (3)^2(4) \\ &= 36 \\ &= 113.10 \text{ ft}^3 \end{aligned}$$

Example 2



The soda can above has volume 355 cm^3 and height 12 cm. What is its diameter?

$$\begin{aligned} \text{Volume} &= r^2h \\ 355 &= r^2(12) \\ \frac{355}{12} &= r^2 \\ 9.41 &= r^2 \\ \text{radius} &= 3.06 \\ \text{diameter} &= 2(3.06) = 6.12 \text{ cm} \end{aligned}$$

Problems

Find the volume of each cylinder.

1. $r = 5 \text{ cm}$
 $h = 10 \text{ cm}$ 2. $r = 7.5 \text{ in.}$
 $h = 8.1 \text{ in.}$ 3. $\text{diameter} = 10 \text{ cm}$
 $h = 5 \text{ cm}$
4. $\text{base area} = 50 \text{ cm}^2$
 $h = 4 \text{ cm}$ 5. $r = 17 \text{ cm}$
 $h = 10 \text{ cm}$ 6. $d = 29 \text{ cm}$
 $h = 13 \text{ cm}$

Find the missing part of each cylinder.

7. If the volume is 5175 ft^3 and the height is 23 ft, find the diameter.
8. If the volume is $26,101.07 \text{ inches}^3$ and the radius is 17.23 inches, find the height.
9. If the circumference is 126 cm and the height is 15 cm, find the volume.

Answers

- | | | |
|--------------------------|---------------------------|-----------------------------|
| 1. 785.40 cm^3 | 2. 1431.39 in^3 | 3. 392.70 cm^3 |
| 4. 200 cm^3 | 5. 9079.20 cm^3 | 6. 8586.76 cm^3 |
| 7. 16.93 ft | 8. 28 inches | 9. $18,950.58 \text{ cm}^3$ |

VOLUME OF A CONE

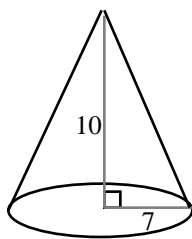
Every cone has a volume that is one-third the volume of the cylinder with the same base and height. To find the volume of a cone, use the same formula as the volume of a cylinder and divide by three. The formula for the volume of a cone of base area A and height h is:

$$V = \frac{A \cdot h}{3} = \frac{r^2 h}{3} \quad \text{or} \quad \frac{1}{3} r^2 h$$

For additional information, see Year 2, Chapter 10, problem MG-26.

Example 1

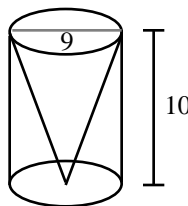
Find the volume of the cone below.



$$\begin{aligned} \text{Volume} &= \frac{1}{3} (7)^2 \cdot 10 \\ &= \frac{490}{3} \\ &= 513.13 \text{ units}^3 \end{aligned}$$

Example 2

Find the volume of the cone below.



$$\begin{aligned} \text{radius} &= 4.5 \\ \text{Volume} &= \frac{1}{3} (4.5)^2 \cdot 10 \\ &= 67.5 \\ &= 212.06 \text{ units}^3 \end{aligned}$$

Example 3

If the volume of a cone is 4325.87 cm^3 and its radius is 9 cm, find its height.

$$\begin{aligned} \text{Volume} &= \frac{1}{3} r^2 h \\ 4325.87 &= \frac{1}{3} (9)^2 \cdot h \\ 12977.61 &= (81) \cdot h \\ \frac{12977.61}{81} &= h \\ 51 \text{ cm} &= h \end{aligned}$$

Problems

Find the volume of each cone.

1. $r = 4$ cm
 $h = 10$ cm

2. $r = 2.5$ in.
 $h = 10.4$ in.

3. $d = 12$ in.
 $h = 6$ in.

4. $d = 9$ cm
 $h = 10$ cm

5. $r = 6\frac{1}{3}$ ft
 $h = 12\frac{1}{2}$ ft

6. $r = 3\frac{1}{4}$ ft
 $h = 6$ ft

Find the missing part of each cone described below.

7. If $V = 1000$ cm³ and $r = 10$ cm, find h .

8. If $V = 2000$ cm³ and $h = 15$ cm, find r .

9. If the circumference of the base = 126 cm and $h = 10$ cm, find the volume.

Answers

1. 167.55 cm³

2. 68.07 in.³

3. 226.19 in.³

4. 212.06 cm³

5. 525.05 ft³

6. 66.37 ft³

7. 9.54 cm

8. 11.28 cm

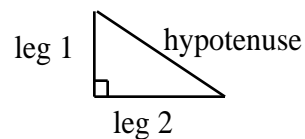
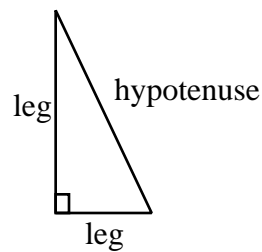
9. 4211.24 cm³

THE PYTHAGOREAN THEOREM

A right triangle is a triangle in which the two shorter sides form a right angle. The shorter sides are called legs. Opposite the right angle is the third and longest side called the hypotenuse.

The Pythagorean Theorem states that for any right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

$$(\text{leg 1})^2 + (\text{leg 2})^2 = (\text{hypotenuse})^2$$

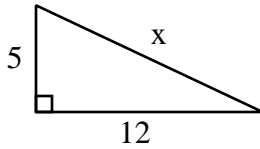


For additional information, see Year 2, Chapter 8, problems GS-1 through 6.

Example 1

Use the Pythagorean Theorem to find x .

a)



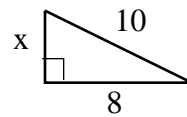
$$5^2 + 12^2 = x^2$$

$$25 + 144 = x^2$$

$$169 = x^2$$

$$13 = x$$

b)



$$x^2 + 8^2 = 10^2$$

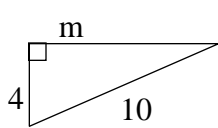
$$x^2 + 64 = 100$$

$$x^2 = 36$$

$$x = 6$$

Example 2

Not all problems will have exact answers. Use square root notation and your calculator.



$$4^2 + m^2 = 10^2$$

$$16 + m^2 = 100$$

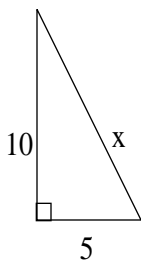
$$m^2 = 84$$

$$m = \sqrt{84} \quad 9.17$$

Example 3

A guy wire is needed to support a tower. The wire is attached to the ground five meters from the base of the tower. How long is the wire if the tower is 10 meters tall?

First draw a diagram to model the problem, then write an equation using the Pythagorean Theorem and solve it.



$$x^2 = 10^2 + 5^2$$

$$x^2 = 100 + 25$$

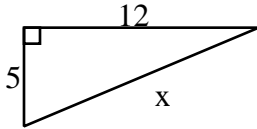
$$x^2 = 125$$

$$x = \sqrt{125} \quad 11.18 \text{ cm}$$

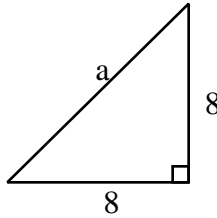
Problems

Write an equation and solve it to find the length of the unknown side. Round answers to the nearest hundredth.

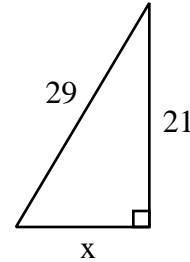
1.



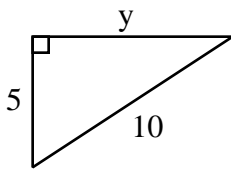
2.



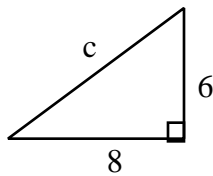
3.



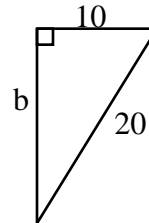
4.



5.



6.



Draw a diagram, write an equation, and solve it. Round answers to nearest hundredth.

- Find the diagonal of a television screen 30 inches wide by 35 inches tall.
- A four meter ladder is one meter from the base of a building. How high up the building will the ladder reach?
- Sam drove eight miles south and then five miles west. How far is he from his starting point?
- The length of the hypotenuse of a right triangle is six centimeters. If one leg is four centimeters, how long is the other leg?
- Find the length of a path that runs diagonally across a 55 yard by 100 yard field.
- How long an umbrella will fit in the bottom of a suitcase 1.5 feet by 2.5 feet?

Answers

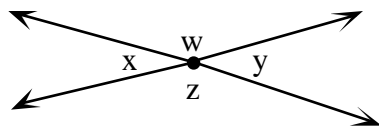
1. 13 2. 11.31 3. 20 4. 8.66 5. 10 6. 17.32
7. 46.10 in. 8. 3.87 in. 9. 9.43 mi 10. 4.47 cm 11. 114.13 yd 12. 2.92 ft

ANGLES, TRIANGLES, AND QUADRILATERALS

PROPERTIES OF ANGLE PAIRS

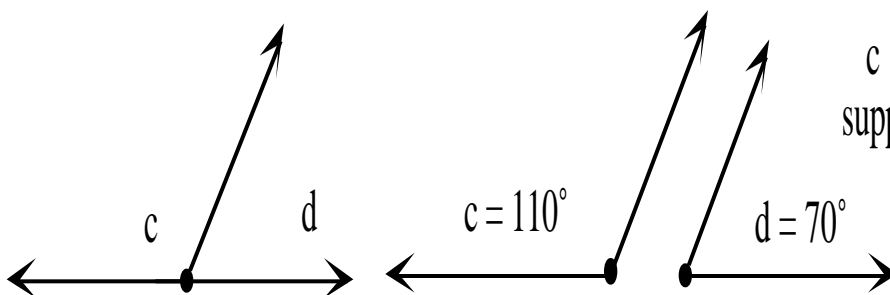
Angles are measured in degrees ($^{\circ}$). A right angle is 90° (clock hands at 3 o'clock). Angles measuring between 0° and 90° are called acute, angles measuring between 90° and 180° are called obtuse.

Intersecting lines form four angles. The pairs of angles across from each other are called vertical angles. The measures of vertical angles are equal.



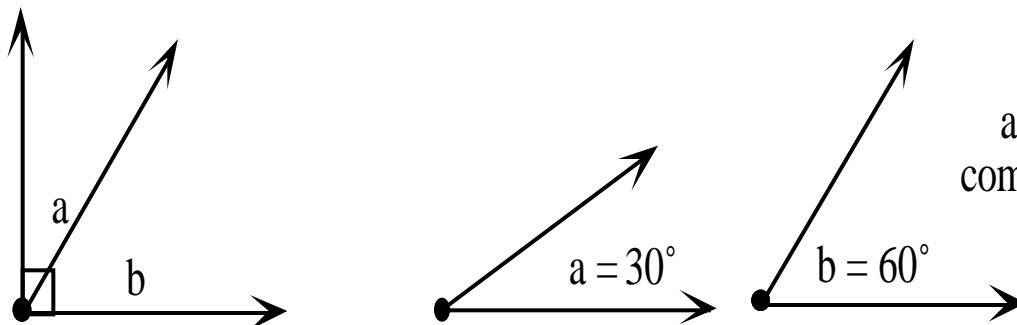
x and y are vertical angles
 w and z are vertical angles

If the sum of the measures of two angles is exactly 180° , then they are called supplementary angles.



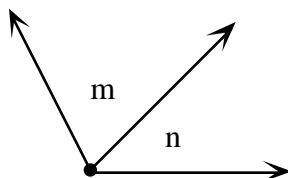
c and d are supplementary

If the sum of the measures of two angles is exactly 90° , then they are called complementary angles.



a and b are complementary

Angles that share a vertex and one side but have no common interior points (that is, do not overlap each other) are called adjacent angles.

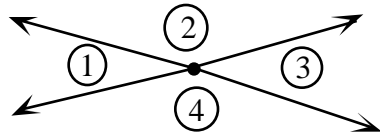


m and n are adjacent angles

For additional information, see Year 1, Chapter 8, problems MC-2, 22, 25, and 42.

Example 1

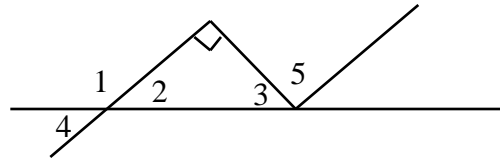
Find the measure of the missing angles if $m \angle 3 = 50^\circ$



- $m \angle 1 = m \angle 3$ (vertical angles)
 $m \angle 1 = 50^\circ$
- $\angle 2$ and $\angle 3$ (supplementary angles)
 $m \angle 2 = 180^\circ - 50^\circ = 130^\circ$
- $m \angle 2 = m \angle 4$ (vertical angles)
 $m \angle 4 = 130^\circ$

Example 2

Classify each pair of angles below as vertical, supplementary, complementary, or adjacent.

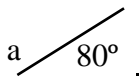


- a) $\angle 1$ and $\angle 2$ are adjacent and supplementary
- b) $\angle 2$ and $\angle 3$ are complementary
- c) $\angle 3$ and $\angle 5$ are adjacent
- d) $\angle 1$ and $\angle 4$ are adjacent and supplementary
- e) $\angle 2$ and $\angle 4$ are vertical

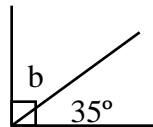
Problems

Find the measure of each angle labeled with a variable.

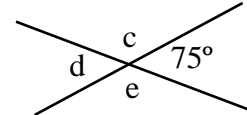
1.



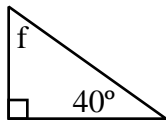
2.



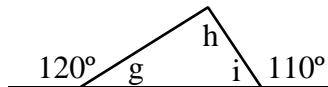
3.



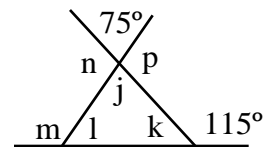
4.



5.



6.



Answers

1. $m \angle a = 100^\circ$

2. $m \angle b = 55^\circ$

3. $m \angle c = 105^\circ$
 $m \angle d = 75^\circ$
 $m \angle e = 105^\circ$

4. $m \angle f = 50^\circ$

5. $m \angle g = 60^\circ$
 $m \angle h = 50^\circ$
 $m \angle i = 70^\circ$

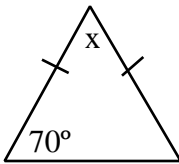
6. $m \angle j = 75^\circ$ $m \angle k = 65^\circ$
 $m \angle l = 40^\circ$ $m \angle m = 140^\circ$
 $m \angle n = 105^\circ$ $m \angle p = 105^\circ$

ISOSCELES AND EQUILATERAL TRIANGLES AND EQUATIONS IN GEOMETRIC CONTEXT

For additional information, see Year 1, Chapter 8, problems MC-53, 58, and 86.

Examples

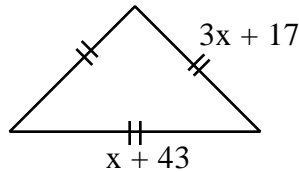
a) Find the measure of x .



Since we have an isosceles triangle, both base angles measure 70° . The sum of the measures of the angles of a triangle is 180° so:

$$\begin{aligned} x + 70^\circ + 70^\circ &= 180^\circ \\ x + 140^\circ &= 180^\circ \\ x &= 40^\circ \end{aligned}$$

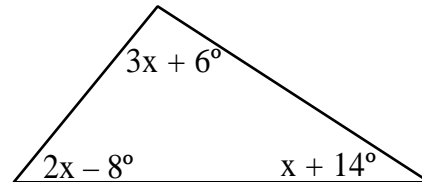
b) Solve for x .



Since we have an equilateral triangle, all sides are of equal length so:

$$\begin{aligned} 3x + 17 &= x + 43 \\ 2x + 17 &= 43 \\ 2x &= 26 \\ x &= 13 \end{aligned}$$

c) Solve for x .



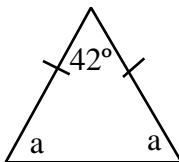
The sum of the measures of the angles of a triangle is 180° so:

$$\begin{aligned} (2x - 8^\circ) + (3x + 6^\circ) + (x + 14^\circ) &= 180^\circ \\ 6x + 12^\circ &= 180^\circ \\ 6x &= 168^\circ \\ x &= 28^\circ \end{aligned}$$

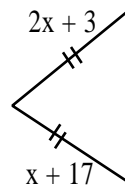
Problems

In each problem, solve for the variable.

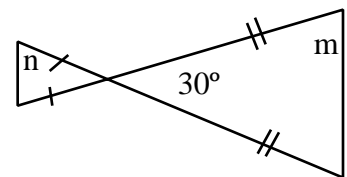
1.



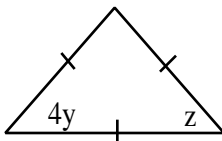
2.



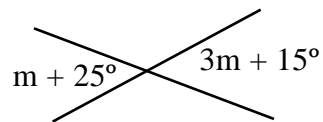
3.



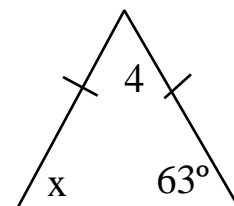
4.



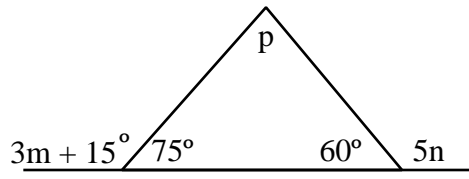
5.



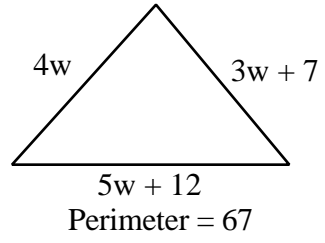
6.



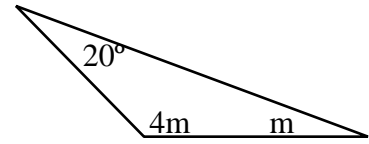
7.



8.



9.



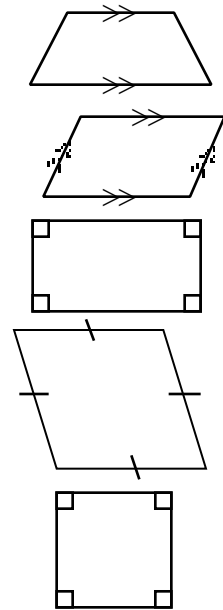
Answers

- | | | |
|---|------------------|---------------------------------|
| 1. $a = 68^\circ$ | 2. $x = 14$ | 3. $m = 75^\circ, n = 75^\circ$ |
| 4. $y = 15^\circ, z = 60^\circ$ | 5. $m = 5^\circ$ | 6. $x = 63^\circ, y = 54^\circ$ |
| 7. $m = 30^\circ, n = 24^\circ, p = 45^\circ$ | 8. $w = 4$ | 9. $m = 32^\circ$ |

QUADRILATERALS

A quadrilateral is any four-sided shape. There are five special cases of quadrilaterals with which students should be familiar.

- Trapezoid – a quadrilateral with one pair of parallel sides.
- Parallelogram – a quadrilateral with two pairs of parallel sides.
- Rectangle – a quadrilateral with four right angles.
- Rhombus – a quadrilateral with four congruent sides.
- Square – a quadrilateral with four right angles **and** four congruent sides.

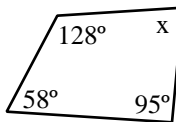


In Year 1, Chapter 8, problems MC-55, 56, and MC-57 students learn that the sum of the measures of the angles in a quadrilateral equals 360° .

Example

Find the measure of x .

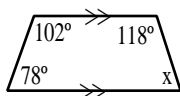
$$\begin{aligned}x + 128^\circ + 95^\circ + 58^\circ &= 360^\circ \\x + 281^\circ &= 360^\circ \\x &= 79^\circ\end{aligned}$$



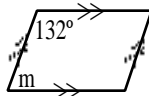
Problems

In each problem solve for the variable.

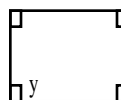
1.



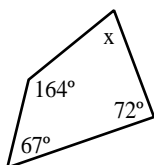
2.



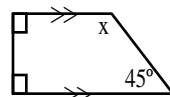
3.



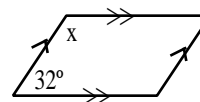
4.



5.



6.



Answers

1. 62° 2. 48° 3. 90° 4. 57° 5. 135° 6. 148°

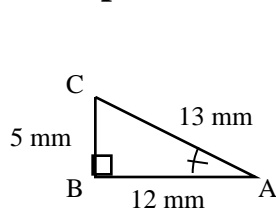
SCALE FACTOR AND RATIOS OF GROWTH

RATIOS OF DIMENSIONAL CHANGE FOR TWO-DIMENSIONAL FIGURES

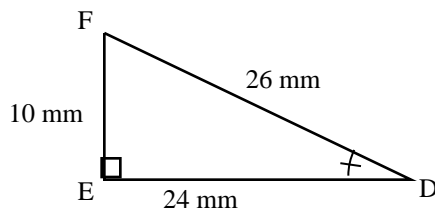
Geometric figures can be reduced or enlarged. When this change happens, every length of the figure is reduced or enlarged equally. For additional information, see Year 2, Chapter 6, problems RS-72, 73, 75 and 88 for examples.

The ratio of any two corresponding sides of the original and new figure is called a scale factor. In this book we always place new figure measurements over their original figure measurements in a scale ratio.

Example



original triangle



new triangle

Side length ratios:

$$\frac{DE}{AB} = \frac{24}{12} = \frac{2}{1}$$

$$\frac{FD}{CA} = \frac{26}{13} = \frac{2}{1}$$

$$\frac{FE}{CB} = \frac{10}{5} = \frac{2}{1}$$

Perimeter ratio:

$$\frac{DEF}{ABC} = \frac{60}{30} = \frac{2}{1}$$

Area ratio:

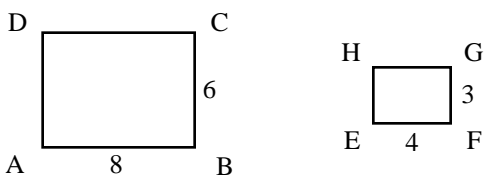
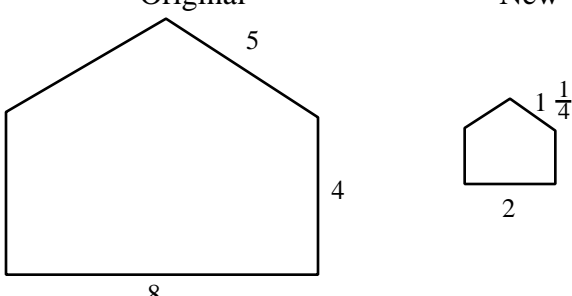
$$\frac{DEF}{ABC} = \frac{120}{30} = \frac{4}{1} \text{ or } \left(\frac{2}{1}\right)^2$$

The scale factor for length is 2 to 1.

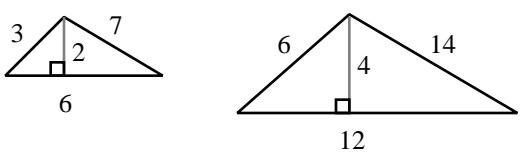
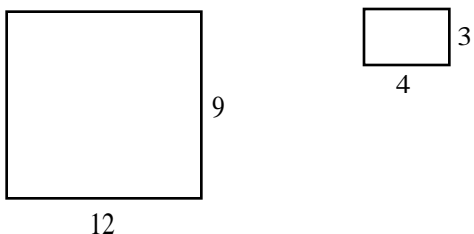
The scale factor for area is 4 to 1.

Problems

Determine the scale factor for each pair of figures below.

1. Original New
- 
2. Original New
- 

Find the perimeter and the area scale factor for each pair of figures below.

3. Original New
- 
4. Original New
- 

5. Two triangles are similar. The new triangle was enlarged by a scale factor of $\frac{3}{1}$. Use a proportion to solve the problems below.
- If the original triangle has a perimeter of 12, what is the perimeter of the new triangle?
 - If the original triangle has an area of 6, what is the area of the new triangle?
6. Two rectangles are similar. The new rectangle was reduced by a scale factor of $\frac{1}{4}$. Use a proportion to solve the problems below.
- If the original rectangle has a perimeter of 24, what is the perimeter of the new rectangle?
 - If the original area is 32 square units, what is the new area?

Answers

1. $\frac{4}{8} = \frac{1}{2}$

2. $\frac{2}{8} = \frac{1}{4}$

3. perimeter scale factor = $\frac{32}{16} = \frac{2}{1}$

4. perimeter scale factor = $\frac{14}{42} = \frac{1}{3}$

area scale factor = $\frac{24}{6} = \frac{4}{1}$

area scale factor = $\frac{12}{108} = \frac{1}{9}$

5. a) 36 units b) 54 sq. units

6. a) 6 units b) 2 sq. units

RATIOS OF DIMENSIONAL CHANGE FOR THREE-DIMENSIONAL OBJECTS

When looking at three-dimensional shapes and enlarging or reducing them, a third dimension (height) is taken into consideration. The volume scale factor then becomes the cube of the scale factor (length ratio).

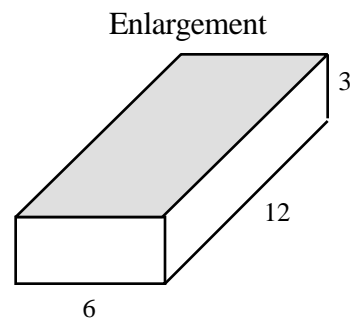
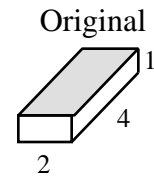
Volume of the original = $2 \cdot 4 \cdot 1 = 8 \text{ units}^3$

Volume of the enlargement = $6 \cdot 12 \cdot 3 = 216 \text{ units}^3$

Scale factor of volume = $\frac{216}{8} = \frac{27}{1}$.

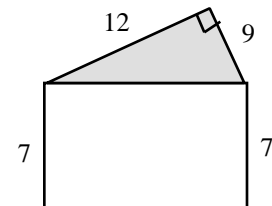
By enlarging each edge by a scale factor of $\frac{3}{1}$, the new volume has a scale factor of $\left(\frac{3}{1}\right)^3 = \frac{3^3}{1^3} = \frac{27}{1}$.

For additional information, see Year 2, Chapter 10, problem MG-14.



Example

Suppose you want to find the surface area and volume of a prism with dimensions three times as large as the figure at right. You could triple the dimensions and do the calculations OR use ratios. Here we use proportions to find the surface area and volume of the new prism after calculating them for the original figure.



- a) Calculate the surface area and volume of the figure at right.

$$B = \frac{12 \cdot 9}{2} = 54 \text{ units}^2, \text{ so } V = 54 \cdot 7 = 378 \text{ units}^3$$

$$\begin{aligned} SA &= 2(54) + 12 \cdot 7 + 9 \cdot 7 + 15 \cdot 7 \quad (\text{Note: } 15 \text{ comes from } \sqrt{12^2 + 9^2} = 15.) \\ &= 108 + 252 \\ &= 360 \text{ units}^2 \end{aligned}$$

- b) Use ratios to find the enlarged figure's surface area and volume.

$$\text{SA: length ratio} = \frac{3}{1} \quad \text{area} = \left(\frac{3}{1}\right)^2 = \frac{9}{1}, \text{ so } \frac{9}{1} = \frac{\text{SA}}{360} \quad \text{SA} = 3240 \text{ units}^2$$

$$\text{V: volume ratio} = \left(\frac{3}{1}\right)^3 = \frac{27}{1} \quad \frac{27}{1} = \frac{\text{V}}{378} \quad \text{V} = 10,206 \text{ units}^3$$

SUMMARY: RATIOS OF DIMENSIONAL CHANGE

For any pair of similar figures or solids with a scale factor $\frac{a}{b}$, the enlargement and reduction relationships are:

Length
(one dimension)

$$\frac{a}{b}$$

Area
(two dimensions)

$$\frac{a^2}{b^2}$$

Volume
(three dimensions)

$$\frac{a^3}{b^3}$$

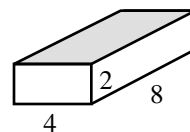
Length ratios apply to sides, edges, and perimeters of figures.

Area ratios apply to surface area of solids and the area of two-dimensional regions.

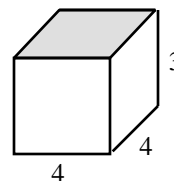
See Year 2, Chapter 10, problem MG-14 for the complete Tool Kit entry about dimensional change.

Problems

1. Reduce the prism at right by a scale factor of $\frac{1}{2}$. Find the original volume and compare it to the new volume. That is, determine $\frac{\text{new volume}}{\text{old volume}}$.



2. The dimensions of a cube are 4 by 4 by 4. Suppose the cube is enlarged by a scale factor of $\frac{2}{1}$. Compare the volumes.



3. If you have a square prism shown at right and you enlarge it by a scale of 5 to 1, what is the new volume?
4. If you reduce a prism with a volume of 81 units^3 by a scale factor of $\frac{1}{3}$, what is the new volume?
5. A prism with an original volume of 24 inches^3 is enlarged. The resulting new volume is 192 inches^3 . What is the scale factor used for enlarging?

Answers

1. The new dimensions are 2 by 4 by 1, so $V = 8 \text{ units}^3$.

$$\frac{8}{64} = \frac{1}{8}$$

2. The new dimensions are 8 by 8 by 8, so $V = 512$.

$$\frac{512}{64} = \frac{8}{1}$$

3. $\text{Volume} = 4 \cdot 4 \cdot 3 = 48 \text{ units}^3$.

Scale factor for volume is $\left(\frac{5}{1}\right)^3 = \frac{125}{1}$.

$$\frac{125}{1} = \frac{V}{48} ; V = 6000 \text{ units}$$

4. $\frac{3}{81} = \frac{1}{27}$; new volume = 3 units^3

5. $\frac{192}{24} = \frac{8}{1}$; scale factor 2 : 1

