UNIT 7

THE BIG RACE

In Unit 7, *The Big Race: Slopes and Rates of Change*, the students will continue to work on understanding the relationship between graphs and their equations. While doing this, they will focus on the components of a line: its slope and intercepts. Intercepts (where lines cross the horizontal and/or vertical axis) usually do not pose a problem for most students, but some students do have trouble with slope. This is usually because they encounter slope as a formula that offers little sense as to what slope represents or means. To avoid this difficulty, the unit stresses that the meaning of slope is **rate of change**. For example, slope will indicate income per month in one problem, then the rate of speed measured by meters per second in the next. Each of these situations represents a rate of change.

After studying slope, students will discover another way to quickly graph a line using the slope-intercept form of an equation. This will also allow them to write the equation of a line given either the slope and y-intercept, a point and the slope, or a point and the y-intercept.

?*Behind the Problem*

Since this is probably the start of a new semester, encourage your child to continue to work hard. There is often a third-quarter slump for many students because finals are over, it is half way through the school year, and summer seems far away. The homework will continually spiral all the types of problems and topics we have studied so far this year. If your child has had a great deal of difficulty with the material, this fact should not discourage her. Rather, she should be anxious to see the problems again so that she can improve. If your child has found most topics and problems relatively easy so far, she should not skip these homework problems. Continued practice and exposure can only deepen her understanding.

This unit begins immediately with a slope problem. If your child has already done this problem successfully, be sure to have her explain it completely to you and rework the problem for reinforcement.

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BR-1. Leslie rides at a constant rate of 2 meters per second. On the resource page provided by your teacher, neatly graph and write an equation in terms of x and y that shows the distance Leslie travels. Let x = time in seconds and y = distance in meters. We do not know how long the race will be so extend your graph appropriately.

Find Leslie's equation and label her line appropriately. Keep this resource page in a safe place. We will use it every day during this unit.

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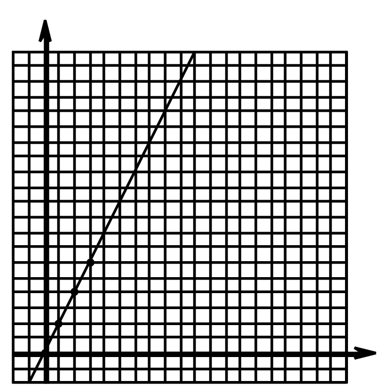
Solution.

At time x = 0 seconds, Leslie hasn't started yet so she has traveled 0 meters. This is the point (0, 0) on the graph. After 1 second, Leslie has traveled 2 meters, since she rides at a constant rate of 2 meters per second. Remember: 2 meters per second means "2 meters <u>every</u> second" or “each second she rides 2 meters.” After 2 seconds, she has ridden 4 meters, and after 3 seconds, 6 meters. These are the points (1, 2), (2, 4), and (3, 6). The relationship between x and y is the equation y = 2x.

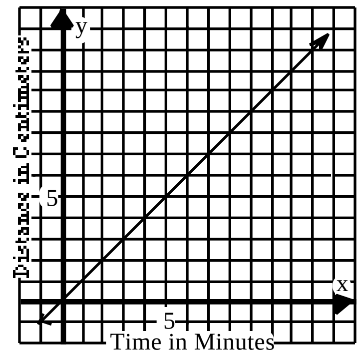


Try it!

1. Miranda's pet snail moves at a rate of 1 centimeter per minute. Graph and write an equation in terms of x and y that shows the distance her snail travels. Let x represent the time in minutes and y represent the distance in centimeters.

Solution.

1. Possible points: (0, 0), (1, 1), (2, 2), (3, 3).
   Equation: y = x



When we use the rate at which Leslie rides her tricycle to graph the line, or the rate the snail moves, we are using the slope of the line to obtain points for graphing.

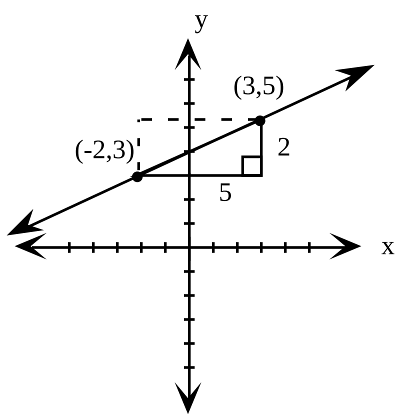
THE SLOPE OF A LINE

The **SLOPE** of a line is a measure of the rate at which something changes. It represents both "steepness" and direction.

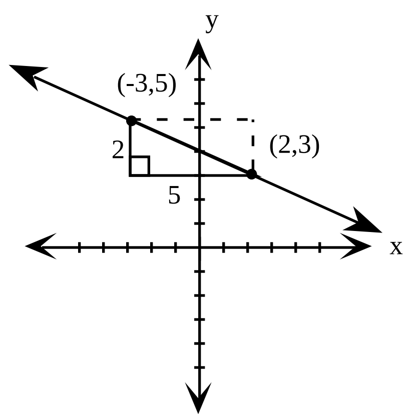
$$\textbf{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{change in y-values}}{\text{change in x-values}}$$

Note that lines that go **upward** from left to right have **positive** slope, while lines that go **downward** from left to right have **negative** slope. The slope of a line is often denoted by the letter "m." Some texts refer to the vertical change as the "rise" and the horizontal change as the "run."

To calculate the slope of a line, graph two points on the line, draw the slope triangle (as shown in the examples), write the ratio, and lastly check to see if the slope is positive or negative.



$m = \frac{2}{5}$



$m = -\frac{2}{5}$

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BR-29. Use the two-point method to graph $y = -3x + 4$.

a) What is the y-intercept?

b) What is the x-intercept?

c) Graph the line and find its slope.

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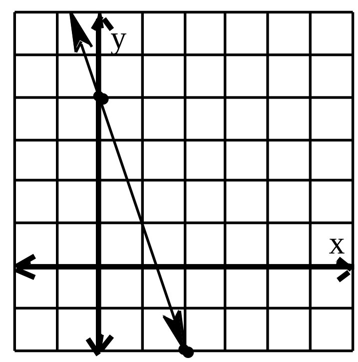
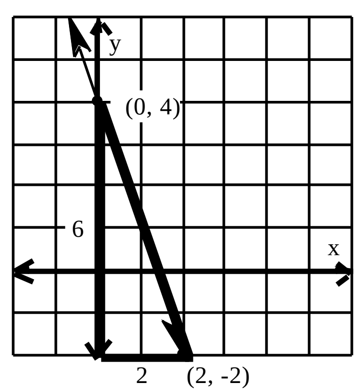


Solution.
With the two-point method, we first plot the y-intercept: (0, 4). Then, choose another value for x, substitute it, and solve for y. This gives another point. For instance, if x = 2, $y = -3(2) + 4 = -6 + 4 = -2$, so we plot the point (2, -2). Connect the points to form the line. The y-intercept is where the graph crosses the y-axis. Here it is the point (0, 4).

It is not so obvious from the graph what the x-intercept is. To calculate it we should substitute y = 0 into the equation and solve for x:

$$0 = -3x + 4$$
$$-4 = -3x$$
$$\frac{-4}{-3} = \frac{-3x}{-3}$$
$$x = \frac{4}{3}$$

So the x-intercept is $\left(\frac{4}{3}, 0\right)$.



We can find the slope of the line by drawing an appropriate right triangle on the graph of the line that will tell us the rate that the line is changing. We can draw any triangle but there are some triangles that will make the problem easier than other triangles. For instance, using the triangle formed by the axes would work, but we would have to work harder to calculate the slope: if we didn't already know the x-intercept we would have to calculate it. Also since the x-intercept is not a whole number, we would have to simplify a messy complex fraction to determine the slope. Using points on the line with whole number coordinates (known as lattice points) will make the task simpler. The triangle shown in the figure at right is such a triangle. For this triangle, the vertical side (change) is 6 and the horizontal side (change) is 2. Since the line goes downward from the left, the slope is negative. Therefore the slope is

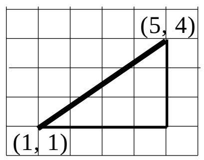
$m = -\frac{6}{2} = \frac{-3}{1}$ or -3.

As your child practices graphing lines and finding their slopes, she will begin to see short cuts. Use the problem below to check whether your child understands how to find the slope of a line.

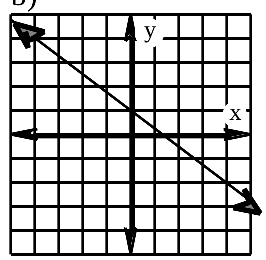
Try it!

2. Calculate the slope for each line below.

a)



b)



c) The line passing through the points (-2, -4) and (5, 1).

Solution.

a) Vertical change is 3, horizontal change is 4, the line goes up (+), so slope = $\frac{3}{4}$.

b) Use (0, 1) and (4, -2) or (0, 1) and (-4, 4) for your lattice points and draw in a right triangle. The vertical change is 3, horizontal change is 4, and the line goes down (–), so slope = $-\frac{3}{4}$.

c) Plot the points on graph paper, sketch a right triangle, and the slope should be $\frac{5}{7}$.

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BR-51. Explain to a student who was absent today how to find the slope and y-intercept of a line in the form $y = mx + b$.

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Solution.

If an equation of a line is written in the form $y = mx + b$, then (as we learned earlier) the y-intercept is the point (0, b). The slope of the line is the coefficient of x, represented in the general form of the equation as m. So in the equation $y = \frac{2}{3}x + 7$, the slope is $\frac{2}{3}$ and the y-intercept is (0, 7).

Try it!

3. What are the slopes of the graphs of the following equations?

a) $y = \frac{7}{8}x - 6$ b) $y = \frac{1}{2}x + \frac{2}{3}$ c) $y = \frac{3}{4} - 8x$

Solution.

a) $\frac{7}{8}$ b) $\frac{1}{2}$ c) -8

Let's first see how to use the information in slope-intercept equations to graph a line.

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BR-76. Without making a table, graph each line. Start with the y-intercept, then use the slope.

a) $y = \frac{2}{3}x - 2$

b) $y = -\frac{2}{3}x + 2$

c) $y = 3x$

d) $y = 4 - 2x$

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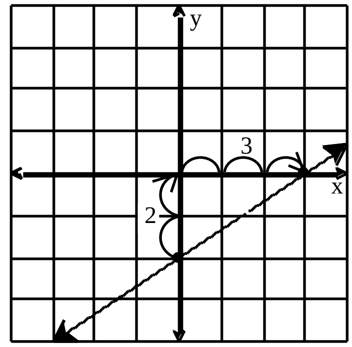
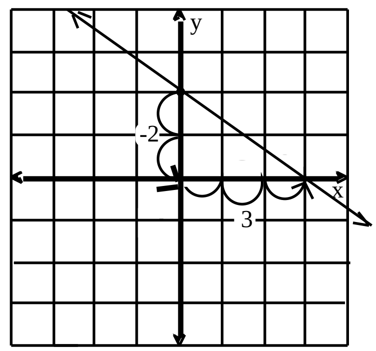
Solution.

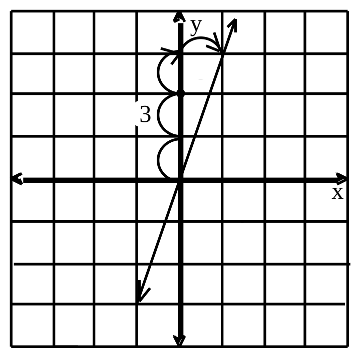
In part (a), we start by identifying the slope and y-intercept. The slope is $\frac{2}{3}$ and the y-intercept is (0, -2). To graph the line we plot the y-intercept. (Before continuing, imagine what the line will look like. You can even encourage your child to lay her pencil down in the general position (i.e., steep or flat and through the y-intercept) of the actual line itself. The fact that the slope is positive tells us the direction of the line is upward left to right.) Then, knowing that the slope is $\frac{2}{3}$, we can find another point on the line by <u>starting</u> at the y-intercept, moving our pen or pencil up vertically two units and then horizontally (left to right) two units. Just remember that the slope is positive! After moving vertically 2 units and horizontally 3 units, we arrive at another point on the line. Two points is enough to graph the line.



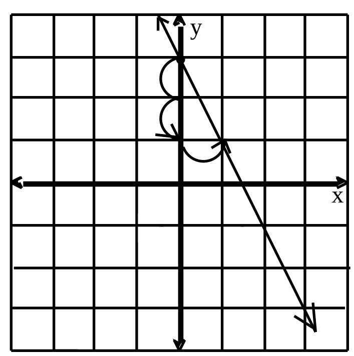
In part (b), $y = -\frac{2}{3}x + 2$, the slope is $-\frac{2}{3}$ and the y-intercept is (0, 2). The slope of $-\frac{2}{3}$ can be thought of as $\frac{-2}{3}$ or $\frac{2}{-3}$ and either one of these equivalent slopes may help you remember (and understand!) how to graph the line when the slope is negative. The result is a line move downward as we view it left to right.



In part (c), $y = 3x$, the slope is 3 and the y-intercept is (0, 0). The slope of 3 can be thought of as $\frac{3}{1}$. Even though the equation might look different, you can still use the patterns from the previous examples. All integer slopes can be written as a ratio with 1 in the denominator. Equations in the form $y = mx$ all pass through (0, 0). Think of those questions this way: if no value appears in the "b" position for $y = mx + b$, then $b = 0$. (0, b) must then be (0, 0).



In part (d), $y = 4 – 2x$, don't let the form of the equation fool you. The slope is -2 or $-\frac{2}{1}$ and the y-intercept is (0, 4). The slope is always the coefficient of x and the y-intercept is always the constant. Rearranging their order doesn't change their meaning.



Try it!

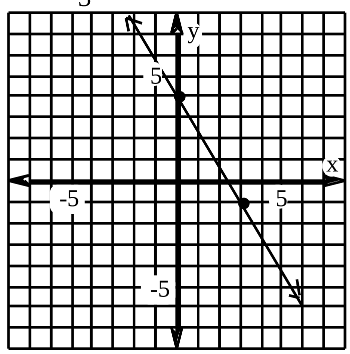
4. <u>Without making a table,</u> graph each line.

a) $y = -\frac{5}{3}x + 4$

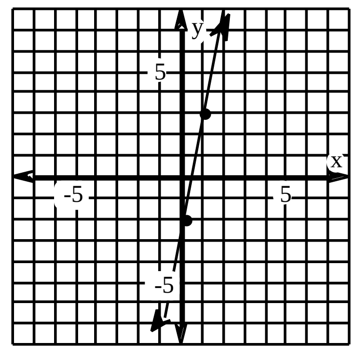
b) $y = 5x - 2$

Solution.

a) $y = -\frac{5}{3}x + 4$

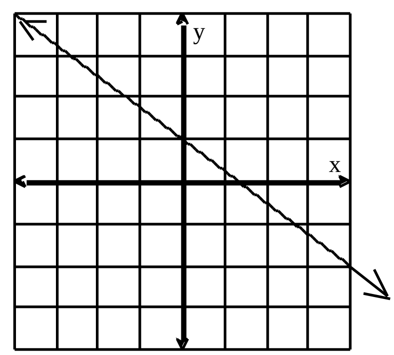


b) $y = 5x - 2$



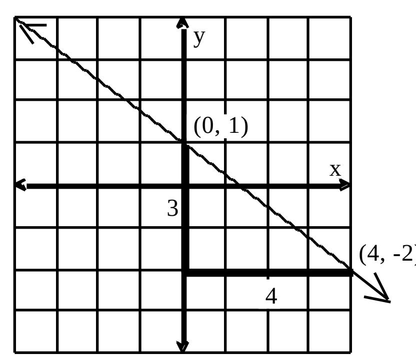
Example.

Write the equation of the line at right.



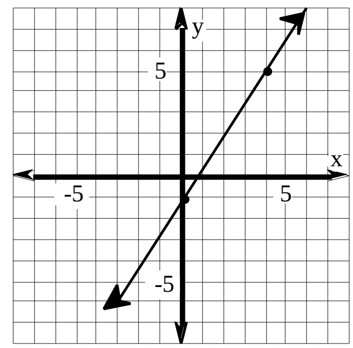
Solution.

We can use the slope and y-intercept to write the equation of the line. In the graph at right, the y-intercept is the point (0, 1) which means the equation of the line must look like y = mx + 1 (since b represents the y-intercept). We draw in an appropriate right triangle (remember to use one with lattice—that is, whole-number—points!) to find the slope is $m = -\frac{3}{4}$. The slope is negative because the line moves downward, left to right. This means we can write the equation as $y = -\frac{3}{4}x + 1$.



Try it!

5. Write the equation of the line shown at right.



Solution.

5. Slope: $\frac{3}{2}$ (use (0, -1) and (2, 2) or (4, 5) or (-2, -4)) ,

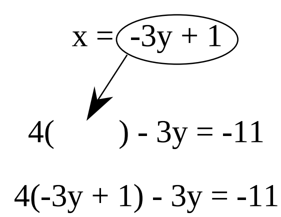
y-intercept (0, -1), equation: $y = \frac{3}{2}x - 1$.

The last topic of importance in this unit is the continuing development of solving systems of equations. In the last unit we saw how we could use graphing or algebra to find the solution. The algebraic method, known as substitution, was illustrated in problems WR-39 through WR-41. We extend this idea in problem BR-71.

MORE ON THE SUBSTITUTION METHOD

In Unit Six, we used Substitution to algebraically find the point of intersection for two linear equations. In those problems, the two equations were always in y-form. However, substitution can be used even if the equations are not in y-form.

Given: $x = -3y + 1$
$4x - 3y = -11$



Use substitution to rewrite the two equations as one:

We can then write: $4(-3y + 1) - 3y = -11$ by replacing x with $(-3y + 1)$.

a) Solve the equation for y.

b) Substitute your answer from part (a) into $x = -3y + 1$. Write your answer for x and y as an ordered pair.

c) Substitute $y = 1$ into $4x - 3y = -11$ to verify that either original equation may be used to find the second coordinate.

d) Why is this method called substitution?

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BR-84. Solve the following systems. Remember to check your solution in both equations to make sure it is the point of intersection.

a) $5x - 4y = 8$
$y = x - 3$

b) $4y - x = 6$
$x = 12 - 2y$

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Solution.

Each of these systems could be solved by graphing, but it would require more work and the potential for errors would increase, especially if the solution involves fractions. To avoid these potential pitfalls, we can use algebra instead. In part (a) we can use the second equation, which tells us what y equals, to replace y in the first equation:

$$5x - 4y = 8$$
$$y = x - 3$$

Substituting $x - 3$ for y in the first equation gives:

$$\begin{aligned} 5x - 4(x - 3) &= 8 \\ 5x - 4x + 12 &= 8 \quad \text{Distributive Property} \\ x + 12 &= 8 \\ x &= -4 \end{aligned}$$

Once we know the value of one variable, we substitute it back into <u>either</u> of the original equations to find the other variable. We can substitute $x = -4$ into either equation, but $y = x - 3$ will require less work to determine the value of y than the other. Using the second equation (try it in both equations if you are not sure why this one will be easier):

$$\begin{aligned} y &= (-4) - 3 \\ y &= -7 \end{aligned}$$

Therefore the solution to the system of equations is (-4, -7). To check, we must show that this point makes **both** equations true. Check:

$5x - 4y = 8$ at (-4, -7):

$$\begin{aligned} 5(-4) - 4(-7) &\overset{?}{=} 8 \\ -20 + 28 &= 8 \quad \text{check!} \end{aligned}$$

$y = x - 3$ at (-4, -7):

$$\begin{aligned} -7 &\overset{?}{=} -4 - 3 \\ -7 &= -7 \quad \text{check!} \end{aligned}$$

Part (b) can be done numerous ways. Just remember that substitutions can occur in many different ways, not just with the y variable.

$$4y - x = 6$$
$$x = 12 - 2y$$

Rather than solving the first equation for y, we can continue by substituting for x. This gives

$$\begin{aligned} 4y - (12 - 2y) &= 6 \quad \text{(Be careful with the negative signs!)} \\ 4y - 12 + 2y &= 6 \\ 6y - 12 &= 6 \\ 6y &= 18 \\ y &= 3 \end{aligned}$$

Now $y = 3$ can be substituted into either of the two original equations. Using the second equation:

$$\begin{aligned} x &= 12 - 2(3) \\ x &= 12 - 6 \\ x &= 6 \end{aligned}$$

The solution is (6, 3). We will leave it to you to check this point in both equations.

Try it!

6. Solve the following systems. Remember to check your solution in both equations to make sure it is the point of intersection.

a)	$y = 3x - 2$		b)	$4x + 5y = 25$
	$2x - y = 7$			$x = 3y - 6.5$

Solution.

6. a) (-5, -17) b) $\left(\frac{5}{2}, 3\right)$

There are many problems in this unit for your child to practice using the slope-intercept form of an equation to graph lines and to write equations. If your child is doing all of the homework problems, he should be seeing them all. Also, don't forget to periodically check your child's Tool Kit. Since he just completed final exams, it should cover the first semester fairly well. Keep adding to it throughout the second semester!

MORE TO TRY

Find the slope of each line.

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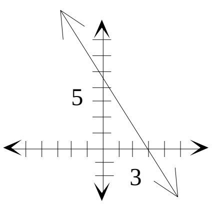

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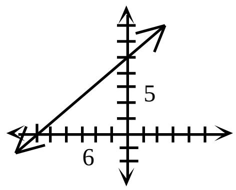

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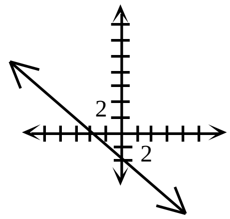

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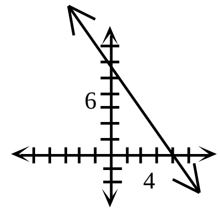

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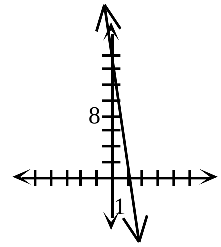

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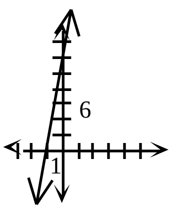

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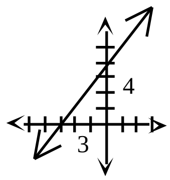

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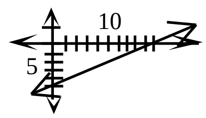

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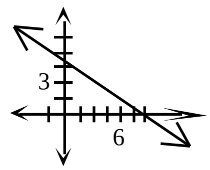

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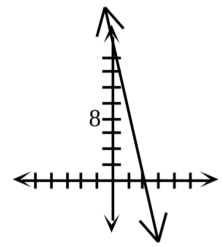


Using two points, graph the following.

11. $y = \frac{3}{5}x + 2$ 12. $y = 4x - 1$ 13. $y = -2x + 5$

Write an equation based upon the Algebra Tiles shown.

14.15.

	x	7
x	x^2	7x
3	3x	21

	2x	5
x	$2x^2$	5x
2	4x	10

Identify the y-intercept and the slope of the graph of each equation.

16. $y = \frac{2}{3}x + 5$ 17. $y = \frac{1}{3}x - 1$ 18. $2y - x = -3$ 19. $y = 3x - 2$

20. $y = \frac{3}{4}x + \frac{3}{4}$ 21. $2x + 3y = 15$ 22. $y = 4x + 5$ 23. $5y - 3x = 10$

Rewrite the following equations in slope-intercept form.

24. $7x + 3 = -2y$ 25. $3x + 2y = 9$ 26. $\frac{7}{x} = \frac{4}{y}$ 27. $6x + 5 = -6y$

28. $4x + 3y = -7$ 29. $\frac{x}{5} = \frac{y}{7}$ 30. $5x - 8 = 3y$ 31. $6x + 5y = 10$

32. $\frac{2}{x} = \frac{3}{y}$

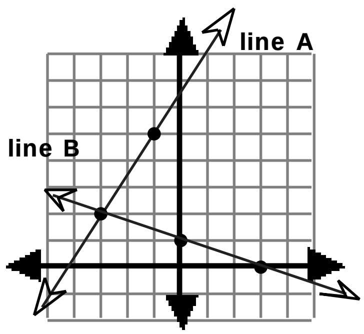
Rewrite each of the following in slope-intercept form.

33. $7y + 14x = 10$ 34. $5x - y = 8$ 35. $5x - 2y = 10$

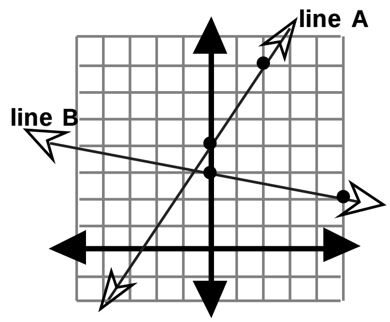
36. $y + 3x = 5$ 37. $2y - 4x = 5$ 38. $3x - y = 6$

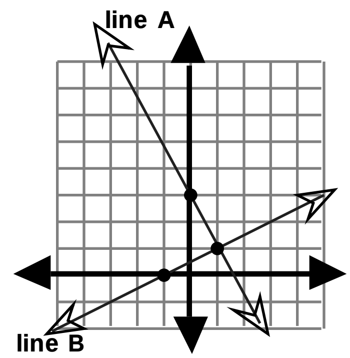
Write an equation for line A and line B.

39.40.

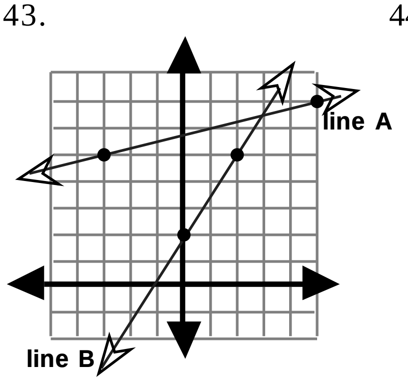


41.

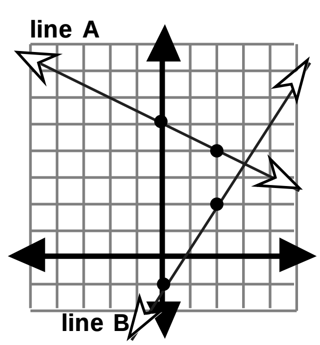


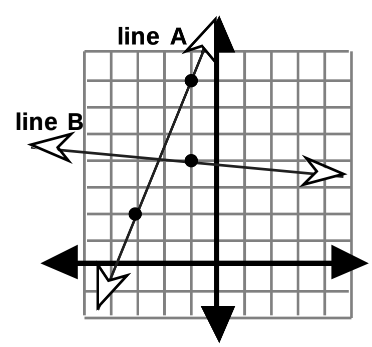


42.43.



44.





Solve the following system of equations by substitution.

45. $y = 3x - 1$
$2x - 3y = 10$

46. $x = -\frac{1}{2}y + 4$
$8x + 3y = 31$

47. $2y = 4x + 10$
$6x + 2y = 10$

48. $y = \frac{3}{5}x - 2$
$y = \frac{x}{10} + 1$

49. $y = -4x + 5$
$y = x$

50. $4x - 3y = -10$
$x = \frac{1}{4}y - 1$

Find the equation for a line with the indicated slope and which passes through the given point.

51. Slope: -3 , (5, 2)

52. Slope: -4 , (0.5, 4)

53. Slope: 6 , (0.333, 8)

54. Slope: 2 , (3, 4)

55. Slope: $\frac{1}{3}$, (6, -2)

56. Slope: $\frac{4}{5}$, (10, 3)

57. Slope: $\frac{-5}{7}$, (21, 13)

58. Slope: $\frac{4}{5}$, (5, -1)

59. Slope: $\frac{6}{7}$, (7, 6)

60. Slope: $\frac{2}{5}$, (8, 11)