CPM Algebra Connections
Mathematical Practices

Introduction
The CPM *Connections* curriculum, developed from 2003-10, mirrors the elements of the CCSS Mathematical Practices. The principles of CPM course design—problem-based lessons, collaborative student work, and spaced practice—are based on the methodological research for teaching mathematics that leads to conceptual understanding. As such, the mathematical practices, similar to previous "best practices" such as the Marzano Principles or CPM's "Ways of Thinking" (see below), are integral to the pedagogy used throughout all of the courses. Task designs ask students to create models, make connections and explain their work regularly. Students are held responsible for high academic rigor, analysis, and critical thinking, and communicate their mathematical findings in writing and/or oral presentations in a clear and convincing manner.

Contents of this resource:

Sample lesson (detailed)
Page 2 presents a detailed review of one lesson from *Algebra Connections* that shows how the eight Mathematical Practices are woven into it.

Selected lessons for review of embedded Mathematical Practices
Page 3 offers two-dozen lessons that the reader can review to see the embedded Mathematical Practices. CPM editors have created a table to indicate which practices are in the lesson and to what degree. The reader should examine the detailed sample lesson on page 2 before examining any of these lessons. Keep in mind that this list is a **sampling** of lessons where you will find the mathematical practices. Elements of the Mathematical Practices are present in most lessons.

CPM's "Ways of Thinking" mirror the Mathematical Practices
The CPM *Connections* courses each focus on five mathematical ways of thinking that are similar or identical to the CCSS Mathematical Practices. Page 4 explains the connection in detail.

Integrating each Mathematical Practice into CPM courses
The paragraphs on pages 5 and 6 discuss in detail how each Mathematical Practice is integrated into the structure of the CPM courses.

Additional resources:
CCSSM Content Standards, Supplemental Lessons, and Pacing Guides
(1) Correlations: There is a separate document that has correlations to the algebra content standards. Be sure to read the list of abbreviations for the coding used in the citations near the top of the first page. **Most references are to lessons in the student textbook.**
(2) Supplement: Note that the other references are all available in the *Algebra Connections CCSSM Supplement* booklet (available February 2011) or via download at the CPM website. These topics provide the additional content—beyond the textbook—required to meet the CCSSM content standards for this course.
(3) Pacing guide: There is also a CCSSM table of contents file for *Algebra Connections* that shows which lessons may be omitted and where the supplementary lessons should be inserted.

Example of how the Mathematical Practices are integrated throughout a lesson in *Algebra Connections*

A typical CPM lesson exemplifies how deeply the CCSS *Standards for Mathematical Practice* are integrated into the course even though the course predates the CCSS practices. In Lesson 3.1.2 in CPM's *Algebra Connections*, "John's Giant Redwood," students work in their collaborative teams conducting an inquiry early in the course into multiple representations of a linear function. In Problem 3-9 students try to find a rule that represents an input-output function table in haphazard order. They do this in an interactive (and fun!) whole-class game (for about 10 minutes) that is set up in a way that when one student finds a solution it does not end the process for the rest of the students. During the game, students need to make sense of the pattern they see, look for additional patterns and structure, and look for consistency in repeated input-output combinations (Practices 1, 7, 8). In Problem 3-10 students reason quantitatively and abstractly (Practice 2) to justify (Practice 3) their finding of the growth rate and their prediction. Students have calculators readily available, but quickly determine that repeated addition on the calculator is not a very efficient process for solving this problem (Practice 5). In Problem 3-11, students model (Practice 4) the growth of the tree with a graph, and start an inquiry into the capabilities of their model; in Problem 3-11(d) students justify (Practice 3) predictions made with the model. Problem 3-12 moves students back and forth between the concrete and abstract (Practice 2) to create an equation that models the situation (Practice 4). Students apply the same skills they used in Problem 3-9(a) from the beginning of the day in their interactive game. In the "Closure" activity for the day (as described in the teacher notes), students discuss the advantages and disadvantages of each of the three representations of the growth function, during which the precision of answers (Practice 6) is discussed. The purpose of the lesson is to begin developing a deep conceptual understanding of rate of growth. Students make sense of growth (Practice 1) by constructing arguments and defending them in collaborative groups, and—toward the end of the day—doing so in a whole-class discussion (Practice 3).

Other Examples for Review—A Partial List

The CCSS *Standards for Mathematical Practice* are regularly integrated into the course design of CPM *Connections*. **The list below is by no means exhaustive (the course has about 135 lessons); it illustrates typical lessons that demonstrate the practices in action. These citations are just a few examples of where the mathematical practices are integrated into the course.**

An "xx" in the table below represents a practice that is a **focus** of the lesson. An "x" represents a practice that is **present** in the lesson.

A FEW EXAMPLES OF THE INTEGRATION OF CCSS PRACTICES INTO THE CPM CURRICULUM

	CCSS Standards for Mathematical Practice							
	1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**
Chapter 2 - Variables and Proportions								
2.1.7, 2.1.8 Recording Work and Solving for *x*		xx			x		x	
Chapter 3 - Graphs and Equations								
3.1.2 John's Giant Redwood	xx	x	x	xx	xx	x	x	x
3.1.3 The Big "C"s	x	x	x		xx		xx	xx
3.1.7 Completing Tables and Drawing Graphs	xx		xx	x		x		
Chapter 4 - Multiple Representations								
4.1.5 Checking The Connections	x	x	x				xx	
4.2.2 Buying Bicycles	xx	xx	x	xx	x			
Chapter 5 - Multiplication and Proportions								
5.1.3 Using Generic Rectangles				x			x	
5.2.1 Setting Up Proportions	x	x	x	x	x		x	x
Chapter 6 - Systems of Equations								
6.2.2 The Hills are Alive	xx	xx	x	xx	x		xx	
6.3.1 Putting it all Together	xx	xx	xx	xx	x		xx	
Chapter 7 - Linear Relationships								
7.1.2 The Roller Coaster	xx	xx	x	xx	xx	x		
7.2.1, 7.2.2, 7.2.3 Slope as Rate	xx	xx	x	xx	xx			
7.3.2 Slopes			xx			x	xx	x
7.3.4 Line Logo Factory	x			x	xx	x	xx	
Chapter 8 - Quadratics								
8.2.2 Water Balloon Contest	xx		xx	x			xx	
8.2.3 Zero Product Property	x	x	xx				xx	
Chapter 9 - Inequalities								
9.1.2 The United Nations	xx	xx	x	xx				
9.2.1 Graphing Two-Variable Inequalities		xx	x				xx	
Chapter 10 - Simplifying and Solving								
10.1.1, 10.1.2 Multiplying/Dividing Rational Expressions					x		xx	xx
10.4.3 Fractional Exponents					x		xx	xx
Chapter 11 - Functions and Relations								
11.1.3 The Cola Machine	xx		x	x			x	
11.2.1 Intercepts and Intersections	xx	xx	x	xx	x	x	x	
Chapter 12 - Algebraic Extensions								
12.1.1 Special Quadratics					x		xx	xx
12.1.2, 12.1.3 Adding/Subtracting Rational Expressions	x	xx			x		xx	xx

CPM's "Ways of Thinking" mirror the Mathematical Practices

Rather than discretely introducing each mathematical practice as a topic to be learned, CPM integrates these practices throughout each lesson. One way these practices are threaded throughout the CPM *Connections* courses is through its focus on mathematical "Ways of Thinking." These Ways of Thinking represent common ways of working mathematically and thus are forms of mathematical practice. Ways of Thinking differ slightly per course due to the different nature of the content, but several are common across courses. For example, since an important mathematical practice is to regularly ask and answer questions such as "How do I know this is true?" and "Is this always true?", Reasoning and Justifying is one of the Ways of Thinking common to most *Connections* courses. Other Ways of Thinking found across multiple *Connections* courses include choosing a strategy, generalizing, visualizing and investigating. Ways of Thinking found in the *Connections* courses are listed below. (MC = *Making Connections* for middle grades, AC = *Algebra Connections,* GC = *Geometry Connections*, and A2C = *Algebra 2 Connections*.)

MC1: comparing, visualizing, describing and explaining, looking for multiple ways of seeing or doing,
and sense making
MC2: generalizing, reasoning and justifying, reversing, choosing a strategy, visualizing
AC: justifying, generalizing, making connections, reversing thinking, and applying and extending
GC: investigating, examining, reasoning and justifying, visualizing, and choosing a strategy/tool
A2C: justifying, generalizing, choosing a strategy, investigating, and reversing

Specifically, many of the mathematical practices proscribed by the Core Content State Standards document directly relate to the Ways of Thinking. For example, "Make sense of problems and persevere in solving them" asks students to engage in a way of thinking captured by *sense making* (MC1), *making connections* (AC), and *investigating* (GC and A2C). The practice "reason abstractly and quantitatively" is represented by the Way of Thinking referred to as *generalizing* (MC2, AC, and A2C) and *comparing* (MC1). The practice "construct viable arguments and critique the reasoning of others" is emphasized with the Ways of Thinking *describing and explaining* (MC1), *reasoning and justifying* (MC2 and GC) and *justifying* (AC and A2C). Finally, the mathematical practice "use appropriate tools strategically" is addressed with the Ways of Thinking *looking for multiple ways of seeing or doing* (MC1), *choosing a strategy* (MC2 and A2C), and *choosing a strategy/tool* (GC).

In addition to encountering prompts in each closure section which require students to reflect on the different ways they used each Way of Thinking throughout the chapter, the text also highlights the regularity of each Way of Thinking by bolding the frequent instances where students are prompted for that form of thinking.

The CPM *Connections* series predates the CCSS *Standards for Mathematical Practice* by several years, yet the practices advocated by the *Standards* are naturally integrated as a core foundation of the CPM curriculum.

Each Standard of Mathematical Practice is Integrated into CPM

Standard 1 of the CCSS *Standards for Mathematical Practice* requires students to **"Make sense of problems and persevere in solving them."** The *Connections* courses have students solve realistic, non-routine problems that are rich in mathematics on a daily basis. These guided investigations are not mere "word problems" that mimic examples of rules. By having students make sense of the problem, rather than being told how to solve a particular kind of problem step-by-step, CPM problems develop deep conceptual understanding of the mathematics, procedural fluency, and perseverance on a daily basis, in addition to teaching and using problem-solving strategies. The curriculum fosters strategic competence and adaptive reasoning in students.

Standard 2 of the CCSS *Standards for Mathematical Practice* requires students to **"Reason abstractly and quantitatively."** In contrast to offering word problems at the end of each chapter, the CPM program generally presents mathematical ideas in contexts *first*, helping students make sense of otherwise abstract principles. Only then do students move on to abstraction and generalization using symbolic notation. Students are taught how to gather and organize information about these contextual problems, break them into smaller parts, look for connections to previous mathematics, and identify patterns and relationships that lead to solutions. Students are also asked to work in reverse, that is, create situations for abstract generalizations.

Standard 3 requires students to **"Construct viable arguments and critique the reasoning of others."** In CPM *Connections* courses, students regularly share information, opinions, and their expertise in collaborative study teams. They work at tables where they have room to manipulate their learning materials and tools. They take turns talking, listening, contributing, arguing, asking for help, checking for understanding, and keeping each other focused. More importantly, during this process students are using higher-order thinking: providing clarification, building on each other's ideas, analyzing and coming to consensus, and productively criticizing. Justifying and critiquing is a part of daily life in a CPM classroom, not an occasional assignment. For each problem, students are expected to communicate their mathematical findings in writing, in oral presentations, or in poster presentations in a clear and convincing manner. Teachers answer students' questions, but do so in a manner that challenges and motivates students to develop and test solutions themselves.

Standard 4 has students **"Model with mathematics."** Modeling contextual situations with multiple representations is a recurring theme in the CPM *Connections* series. For example, from their earliest work with proportions and linear functions all the way through the more complex functions of later courses, students consistently model functions using tables, graphs, equations, and narrative or diagrams. In creating these models, students make assumptions, then predictions, and then check to see if their predictions make sense in the context of the problem. Students regularly use area models to multiply fractions, multiply and divide polynomials, factor, and solve probability problems. In contexts involving variability in data, students learn that a model may not be perfect, yet can be very useful for describing data and making predictions. CPM students find that a calculator or computer can help them model repeated probabilistic experiments much more efficiently than actually conducting the experiment.

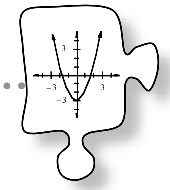
Standard 5 requires students to **"Use appropriate tools strategically."** In the typical CPM lesson, students have available to them a cornucopia of tools—from rulers and scissors, to tracing paper and graph paper, to blocks and tiles, to calculators—but are not typically told which specific tools to use to solve any particular problem. Indeed, a team of collaborating CPM students usually has a designated Resource Manager, whose task it is to ask the teacher for the tools their team needs for that lesson. It is not unusual for different teams to use different tools to solve a problem; during the course of the lesson students share with the whole class their solution strategies, and frequently this includes a lively discussion of which tools were most efficient and productive to solve a given problem. For problems where students are becoming fluent with algebraic procedures, the CPM *Connections* texts use an icon to indicate calculators should not be used. But during investigations students may choose to explore with their calculators to make sense of the mathematics without getting bogged down in computations. During various lessons, students might be exploring in a computer lab with programs provided by CPM, using motion detectors to determine rates, or using lasers or computer-based applets to demonstrate a point.

Standard 6 requires students **"Attend to precision."** Since they are solving contextual problems on a daily basis, the need for attending to precision soon becomes a natural consequence of being a CPM student. Whether they are converting the units in a problem to be consistent, or checking whether a numerical solution makes sense, dealing with precision in choosing units is inherent. Many CPM investigations make use of a calculator; using calculators extensively requires students to frequently attend to the precision of the results displayed. Since most problems are contextual, when students are symbolically solving problems, the mantra of "defining variables with units" becomes essential to coming up with a solution that makes sense. In the case of trigonometric or exponential situations, problems often require decimal approximations to make sense of the solution; CPM students find that approximations made in these situations may require higher levels of precision when evaluating expressions. They also determine that four decimal places of precision is useless when measuring angles in a garden plot.

Standard 7 requires students to **"Look for and making use of structure."** Since CPM students are developing deep conceptual understanding of the underlying mathematics, they frequently use this practice to bring closure to investigations. For example, cross-multiplying to find equivalent fractions is not taught simply as a procedure to be practiced, but is developed from the underlying structure of a multiplication table. Students develop deep conceptual connections between proportions, growth, steepness, and slope by exploring different manifestations of the structure of rates. CPM students do not simplify rational expressions by "canceling;" instead they use the underlying structure of the "Giant One"—fractions where the numerator and denominator are equal. Theorems in geometry are developed from the structure of repeated translations, not just listed in isolation. Polynomials are not multiplied and divided by following an algorithm, but by looking at the underlying structure of an area model. Moreover, polynomials are not solved by just following algorithms, but by looking at the structure of the factored form and the different kinds of roots that structure leads to.

Standard 8 requires students to, **"Look for and express regularity in repeated reasoning."** When faced with a new investigation of a mathematical concept, CPM students often look for a simpler or analogous problem. By extending the structure of previous problems, students are continually expanding their ability to solve increasingly complex problems. At first students use repeated reasoning in multiplication tables to multiply fractions or find equivalent fractions. Students expand the reasoning of simpler intuitive probability problems into increasingly more complex probabilistic situations. CPM students observe repeated structure in area models and leverage that into the ability to multiply, factor, and eventually divide, polynomials. Students use repeated patterns to make sense of negative, zero, and fractional exponents, and to solve rational expressions. Repeated reasoning allows for increasingly complex geometric proofs to be developed from simpler ones and, more generally, by repeated building on conceptual understanding of previous underlying mathematics, make connections to continually and increasingly more complex situations.

3.1.2 How can I make a prediction?



Using Tables, Graphs, and Rules to Make Predictions

In Lesson 3.1.1, you wrote rules for patterns found in $x \rightarrow y$ tables. In this lesson, you will focus on using variables to write algebraic rules for patterns and contextual situations. You will use a graph to help predict the output for fractional x-values and will then use a rule to predict the output when the input is too large and does not appear on the graph.

While working today, focus on these questions:

How can you write the rule without words?

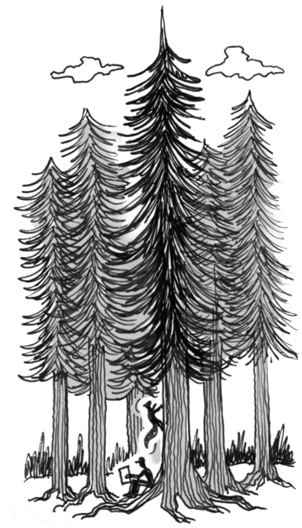
What does x represent?

How can you make a prediction?

3-9. SILENT BOARD GAME

During Lesson 3.1.1, you created written rules for patterns that had no tiles or numbers. You will now write algebraic rules using a table of jumbled in/out numbers. Focus on finding patterns and writing rules as you play the Silent Board Game. Your teacher will put an incomplete $x \rightarrow y$ table on the overhead or board. Study the input and output values and look for a pattern. Then write the rule in words and symbols that finds each y-value from its x-value.

3-10. JOHN'S GIANT REDWOOD, Part One



John found the data in the table below about his favorite redwood tree. He wondered if he could use it to predict the height of the tree at other points of time. Consider this as you analyze the data and answer the questions below. Be ready to share (and **justify**) your answers with the class.

Number of Years after Planting	3	4	5
Height of Tree (in feet)	17	21	25

a. How tall was the tree 2 years after it was planted? What about 7 years after it was planted? How do you know?

b. How tall was the tree the year it was planted?

c. Estimate the height of the tree 50 years after it was planted. How did you make your prediction?

3-11. John decided to find out more about his favorite redwood tree by graphing the data.

a. On the Lesson 3.1.2B Resource Page provided by your teacher, plot the points that represent the height of the tree over time. What does the graph look like?

b. Does it make sense to connect the points? Explain your thinking.

c. According to the graph, what was the height of the tree 1.5 years after it was planted?

d. Can you use your graph to predict the height of the redwood tree 20 years after it was planted? Why or why not?

3-12. John is still not satisfied. He wants to be able to predict the height of the tree at any time after it was planted.

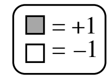
a. Find John's table on your resource page and **extend** it to include the height of the tree in the 0^{th} year, 1^{st} year, 2^{nd} year, and 6^{th} year.

b. If you have not already, use the ideas from the Silent Board Game to write an algebraic rule for the data in your table. Be sure to work with your team and check that the rule works for all of the data.

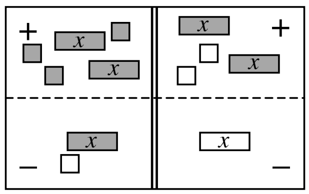
c. Use your rule to check your prediction in part (c) of problem 3-10 for how tall the tree will be in its 50^{th} year. How close was your prediction?

Review & Preview

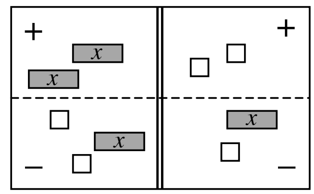
3-13. Write the equation represented in each diagram below on your paper. For each part, simplify as much as possible and then solve for x. Be sure to record your work on your paper.



a.



b.



3-14. Evaluate the following expressions given the values below.

a. $ab + bc + ac$ for $a = 2$, $b = 5$, and $c = 3$

b. $\frac{20-x^2}{y-x}$ for $x = -2$ and $y = 6$

3-15. Copy and simplify the following expressions by combining like terms.

a. $x + 3x - 3 + 2x^2 + 8 - 5x$

b. $2x + 4y^2 - 6y^2 - 9 + 1 - x + 3x$

c. $2x^2 + 30y - 3y^2 + 4xy - 14 - x$

d. $20 + 3xy - 3xy + y^2 + 10 - y^2$

3-16. Use the order of operations to simplify the following expressions.

a. $5 - 2 \cdot 3^2$

b. $(-2)^2$

c. $18 \div 3 \cdot 6$

d. -2^2

e. $(5 - 3)(5 + 3)$

f. $24 \cdot \frac{1}{4} \div -2$

g. Why are your answers for parts (b) and (d) different?

3-17. Mrs. Swanson gives out only one type of candy for Halloween. The local discount store sells six pounds of butterscotch candies for $7.50. Use proportional reasoning to determine the information below. Be sure to explain your answer and organize your reasoning.

a. What is the cost of 18 pounds of butterscotch candies?

b. What is the cost of 10 pounds of butterscotch candies?