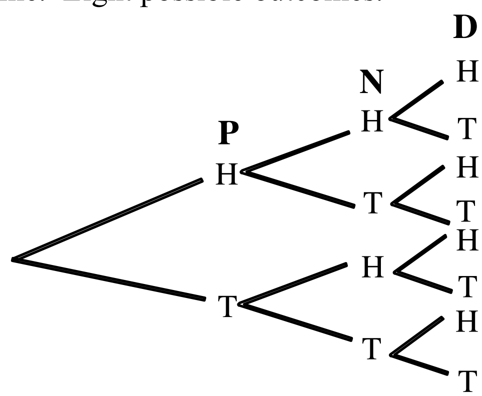
COUNTING PROBLEMS

FUNDAMENTAL COUNTING PRINCIPAL

Suppose that you are flipping three coins: a penny, a nickel, and a dime. How many possible outcomes are there? One way to find the total number of outcomes would be to make an **organized list**: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT where the first letter is the penny, the second is the nickel and the third is the dime. Eight possible outcomes.

Another way to see your options is to make a **tree diagram**. Tree diagrams are usually drawn horizontally. That is, each stage of the event is shown from left to right. Tree diagrams can be done with 2, 3, 4, or more possibilities.



Again, consider flipping the penny, the nickel, and the dime. Start with the penny and show the two possible outcomes (shown under the "P" at right). Since there were two possible outcomes for the penny, there are two possible outcomes for the nickel for <u>each</u> of the penny's outcomes. At this point, the tree diagram shows four possible outcomes for the flip of a penny followed by the flip of a nickel (under the "N").
Next flip the dime. Each flip is **independent** of the other coin flips. If you flip heads on the penny and heads on the nickel, you could get heads or tails on the dime. By following each branch in the diagram, starting from the penny, you can see that there are eight possible outcomes when you flip three coins (shown under the "D").
By following the branches from the start, far left, to the end of the branch, far right, you again find eight possible outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT.

In many cases, for example the number of possible car license plates, making a list or drawing a tree diagram is not practical. However, thinking about the number of choices or branches will allow us to calculate the total number of possible outcomes without actually creating the list. To find the number of choices in the previous coin problem we use a decision chart to organize and find the number of choices:

_____	_____	_____	*multiplying the*	<u>2</u>	<u>2</u>	<u>2</u>	= 8 total choices
penny	nickel	dime	*number of choices*	penny	nickel	dime	
H/T	H/T	H/T		H/T	H/T	H/T	

This is an example of the **Fundamental Counting Principal** which states that if there are *m* choices for event A and *n* choices for event B, then event A followed by event B has $m \cdot n$ different choices.

Example 1 In some states, the license plate of a car consists of three letters followed by three digits. If **repetition is not allowed**, how many possibilities are there?

Make a decision chart:

26	25	24	10	9	8	= 11,232,000 possible license plates
1st letter	2nd letter	3rd letter	1st digit	2nd digit	3rd digit	

How many license plates are possible if **repetition is allowed**?

26	26	26	10	10	10	= 17,576,000
1st letter	2nd letter	3rd letter	1st digit	2nd digit	3rd digit	

Example 2 Before the start of vacation, Gracie went on a shopping spree and bought some new clothes. She bought two pairs of sandals, five tops, and three pairs of shorts. How many different outfits can she create using only her new clothes?

Make a decision chart:

2	3	5	= 30 new outfits
# sandals	# shorts	# tops	

Problems

1. You have decided to take a vacation. You want to go from Los Angeles to San Francisco and then to Hawaii, and you have all summer for your trip. To get from L.A. to San Francisco you can choose to drive, fly, take a bus, or take the train. From San Francisco to Hawaii you can fly, cruise, or sail. In how many different ways can you travel from Los Angeles to Hawaii?

2. While playing Scrabble®, you need to make a word out of the letters A N P S.
   How many arrangements of these letters are possible?

3. How many four-digit numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7 if it is okay to repeat a digit in a number?"

4. How many four-digit numbers could you make with the digits 1, 2, 3, 4, 5, 6, 7 if you could not repeat the use of any digit?

5 A child's game contains nine discs, each with one of the numbers 1, 2, 3, …, 9 on it. How many different 3-digit numbers could be formed by choosing any three discs?

6. A new lotto game called Quick Spin has three wheels, each with the numbers 1, 2, 3, …, 9 equally spaced around the rim. Each wheel is spun once and the number the arrow points to is recorded. How many three digit numbers are possible?

7. How many different ways can you have your quarter pound hamburger prepared if you can have it prepared with or without mustard, ketchup, mayonnaise, lettuce, tomatoes, pickles, cheese, and onions?

8. How many arrangements of the letters in the word SQUARE begin with SQ ?

9. Five teenagers go to the movies. They want to sit together in a row with a student on each aisle (assume the row is 5 seats wide including 2 aisle seats).

   a. How many ways can they sit in the row?

   b. If Kris wants to sit in an aisle seat, how many ways can they all sit in the row?

   c. If Patrice wants to sit on an aisle seat with Beth next to her, how many ways can the five students sit?

10. How many arrangements of all the letters in the word PYRAMID do not end with D?

FACTORIALS

FACTORIAL is a shorthand for multiplication of a list of consecutive, descending whole numbers: $n! = n(n - 1)(n - 2) \bullet \bullet \bullet (3)(2)(1)$. Also, to make some formulas work, $0! = 1$.

For example, 4 factorial = $4 \bullet 3 \bullet 2 \bullet 1 = 24$ and $6! = 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 720$.

Scientific calculators have a factorial (!) key. The answer to many counting problems involve factorials.

Example Marcos is selecting classes for next year. He plans to take English, physics, government, pre-calculus, Spanish and journalism. His school has a six-period day. In how many different ways might the school arrange his day? Using a decision chart:

6	5	4	3	2	1	= 6! or 720 possible schedules
period 1 choices	period 2 choices	period 3 choices	period 4 choices	period 5 choices	period 6 choices	

Remembering what n! means can help you do some messy calculations quickly, as well as help you do problems that might be too large for your calculator.

Example If we wanted to calculate $\frac{9!}{6!}$, we could use the n! button on our calculator and find that 9! = 362880, and 6! = 720, so $\frac{9!}{6!} = \frac{362880}{720} = 504$.

But, if we remember that $9! = 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$ and $6! = 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$ then we can write:

$$\frac{9 \bullet 8 \bullet 7 \bullet \cancel{6} \bullet \cancel{5} \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{\cancel{6} \bullet \cancel{5} \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}} = 9 \bullet 8 \bullet 7 = 504$$

Problems

Write each problem in factorial form and give the answer.

11. What is the possible number of ways to arrange the letters **M A T H**?

12. How many distinguishable batting orders can be made from the nine starting players on a baseball team?

13. How many distinct arrangements of the letters in the word FRACTIONS are there?

14. Five students are running for Junior Class President. They must give speeches before the election committee. In how many different orders could they give their speeches?

15. In how many ways can eight students line up to have their picture taken?

16. Use the simplifying method shown in the last example to simply each of the following before computing.

    a. $\frac{10!}{8!}$ b. $\frac{20!}{18!2!}$

    c. $\frac{7!}{4!3!}$ d. $\frac{75!}{73!}$

PERMUTATIONS

Problems that ask how many arrangements are possible from a set of items without repetition are called **permutations**. Using decision charts or factorials can solve permutation problems.

Example Eight people are running a race. In how many different ways can they come in first, second, and third?

This is a problem of counting permutations, and the result can be represented $_{8}\mathbf{P}_{3}$, which means the number of ways to **choose** **and** **arrange** three things from a set of eight. Using a decision chart:

$$_{8}\mathbf{P}_{3} = \underline{8} \cdot \underline{7} \cdot \underline{6} = 336.$$

A more complicated but more compact way to write this is:

$$_{8}\mathbf{P}_{3} = 8 \cdot 7 \cdot 6 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8!}{5!} = \frac{8!}{(8-3)!}$$

In general, $_{n}\mathbf{P}_{r} = \frac{n!}{(n-r)!}$ for n items chosen r at a time.

Problems

17. Find the value of each permutation.

    a. ${}_{5}\mathbf{P}_{3}$ b. ${}_{7}\mathbf{P}_{4}$ c. ${}_{8}\mathbf{P}_{2}$

18. You finally managed to hold on to some money long enough to open a savings account at the credit union. When you go in to open the account, the accounts manager tells you that you need to select a 4-digit **pin** (**p**ersonal **i**dentification **n**umber). She also says that you can't repeat a digit but that you can use any of the digits 0, 1, 2,..., 9 for any place in your 4-digit code. How many pins are possible?

19. Twenty-five art students submitted sculptures to be judged at the county fair. Awards are going to be given for the six best sculptures. You have been asked to be the judge. You must **choose and arrange in order** the best six sculptures. How many ways are possible?

20. How many five-letter "words" can be formed from the letters in the word COMBINE?

21. Fifty-two contestants are vying for the Miss Teen pageant. In how many different ways can the judges pick the next Miss Teen and her three runners-up?

22. The volleyball team is sponsoring a mixed-doubles sand court volleyball tournament and sixteen pairs have signed up for the chance to win one of the seven trophies and cash prizes. In how many different ways can the teams be chosen and arranged for the top seven slots?

23. You are getting a new locker at school, so the first thing you must do is decide on a new combination. The three number combination can be picked from the numbers 0-35. How many different locker combinations could you make up if none of the numbers can be repeated?

24. Your mother has installed a shelf in your room to display some of your trophies. After you find the box in your closet, you count 15 trophies but the shelf only has space for eight trophies. How many display arrangements are possible?

25. In how many ways can six different algebra books and three different geometry books be arranged on a shelf if all the books of one subject must remain together?

Answers

1.	12	2.	24	3.	2401	4.	840
5.	504	6.	729	7.	256	8.	24
9a.	120	9b.	48	9c.	12	10.	4320
11.	4! = 24	12.	9! = 362,880	13.	9! = 362,880	14.	5! = 120
15.	8! = 40,320	16a.	90	16b.	190	16c.	35
16d.	5550	17.	60	18.	5040	19.	127,512,000
20.	3360	21.	6,497,400	22.	57,657,600	23.	42,840
24.	259,459,200			25.	8640		